# Multidimensional and three-particle scattering problems in the adiabatic representation

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A discussion is given in the adiabatic representation of a consistent formulation of the multidimensional and three-particle inverse scattering problems on the basis of a consistent solution of two interrelated problems: the parametric problem for the Hamiltonian of the fast motion, and the problem for multichannel systems of coupled gauge equations describing the slow dynamics. A method is presented for constructing a large class of exactly solvable multidimensional models by a generalization of the technique of Bargmann potentials to a parametric family of inverse problems and for systems of equations with a covariant derivative. The introduction of an additional matrix of scalar potentials without loss of supersymmetry and, accordingly, the existence of conditions for topological effects are discussed. A natural generalization of Witten's construction of one-dimensional supersymmetric quantum mechanics to systems of gauge equations in two-dimensional space is proposed.

## 1. INTRODUCTION

In our view, the adiabatic representation[1–3] offers the best prospects for solving complicated multidimensional many- and few-particle problems of quantum mechanics. It enables one to take into account the mutual influence of the slowly varying external field and a rapidly varying internal field. In the adiabatic approach, the coupled slow, $s$, and fast, $f$, subsystems are separated. Accordingly, the total Hamiltonian is divided into constituent parts: $H=h^s+h^f$. Fast subsystems can be assumed to be embedded in slow subsystems, which, in their turn, influence the properties of the fast systems. The solution of the complete scattering problem in this case reduces to a multichannel problem that describes the slow dynamics of quantum systems and to a single-channel (for the Hamiltonian) fast motion that depends parametrically on the slow coordinate variables. The total wave function of the system is represented in the form of an expansion

$$\Psi(\mathbf{X})=\sum \int \Phi_n(\mathbf{X};.)\chi_n(\mathbf{X})$$

with respect to the eigenstates $\Phi_n(\mathbf{X};.)$ of the Hamiltonian $h^f(\mathbf{X};.)$ of the fast motion for each fixed set of values of the slow variables $\mathbf{X}$. The period denotes the fast variables. Substitution of such an expansion of $\Psi$ in the original multidimensional Schrödinger equation and averaging over the fast variables of the internal motion leads to a multichannel scattering problem for the slow subsystems with covariant derivative $D(\mathbf{X})=I\otimes\nabla-i\mathbf{A}(\mathbf{X})$. In contrast to ordinary multichannel scattering theory, in which the coupling between the channels is determined by the matrix elements of the potential energy, in the multichannel equations of the adiabatic approach the coupling between the states of the Hamiltonian of the fast motion is realized by the matrix elements of the induced connection operator $\mathbf{AX}=i\langle\Phi(\mathbf{X};.)|\nabla|\Phi(\mathbf{X};.)\rangle$.

In Refs. 4–14 direct and inverse problems were formulated for systems of equations of gauge type, and also a parametric inverse problem for the Hamiltonian of the fast motion. Simultaneous solution of the two problems gives the complete solution to the problem. In the adiabatic representation, the multichannel inverse problem reduces to determination of the $S$ matrix from known multidimensional amplitudes $f(\mathbf{P},\hat{\mathbf{X}})$ and subsequent reconstruction of the effective vector, $\mathbf{A}$, and potential, $U$, matrices, and also matrix solutions $\chi$. Essentially, it is solved by using a unitary transformation $\mathscr{U}$ of gauge type that realizes a transition to a fixed basis and reduces the system with a covariant derivative to a system of ordinary equations with coupling through an effective potential matrix. In this way the prerequisites were created for generating a large class of exactly solvable models for multidimensional and many-particle objects. However, the problem is not fully solved until one has determined the original multidimensional potential $\mathbf{V}^f(\mathbf{X},\mathbf{Y})$ that describes the fast dynamics. It cannot be determined as in ordinary multichannel scattering theory even after the potential matrix elements $U^f=\langle\Phi|V^f|\Phi\rangle$ have been found from given $S$ matrix. The difficulty is that the basis functions $|\Phi(\mathbf{X},\mathbf{Y})\rangle$ are determined by the same potential $V^f(\mathbf{X},\mathbf{Y})$ in the solution of the "parametric" equation, and this is, in general, not known in advance. One can recover the potential $V^f(\mathbf{X},\mathbf{Y})$ and find moving-frame functions $|\Phi(\mathbf{X};\mathbf{Y})\rangle$ by means of the formalism of the inverse problem for the fast equation with parametric dependence of the scattering data on the slow variables $s(\mathbf{X};\mathbf{k})$, $\mathscr{E}_n(\mathbf{X})$, $\gamma_n(\mathbf{X})$, this dependence being determined, in its turn, by solution of the "slow" system of equations.

This is a second important aspect of the adiabatic representation, and it is of interest in connection with the new formulation of the inverse problem with parametric dependence on the slow coordinate variables. To a certain degree, the situation is analogous to the theory of nonlinear

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evolution equations. However, it should be noted that instead of the simple evolution equation in soliton theory it is here necessary to handle the much more complicated system of equations (6) for the determination of the parametric dependence of the spectral data on the slow variables. However, it is true that in this theory there exists an analog of an evolution equation—the bilocal unitary operator (13) of parallel transport of the frame from one point of the base to another.

On the basis of the multichannel and single-channel technique of Bargmann potentials for the slow system of equations and parametrically dependent fast basis equation, there was proposed in Ref. 4 a method of analytical modeling of effective interactions in multidimensional fields and determination of the corresponding solutions.

*One of the interesting aspects of the adiabatic representation is associated with the appearance of gauge fields in nonrelativistic few-particle and many-particle quantum systems, especially in connection with Berry's discovery*[15] *of a geometric adiabatic phase.* Further stages in the development of the theory were marked by the studies of Wilczek and Zee[16] and of Aharanov and Anandan.[17] Wilczek and Zee showed that non-Abelian effective gauge fields arise in the adiabatic treatment of molecular systems with degenerate electron states. Aharanov and Anandan generalized the approach by introducing nonadiabatic non-Abelian geometric phases. One of the consequences of the approach is the occurrence in molecular systems of effects equivalent to the Aharanov–Bohm[18,19] and Hall[20,21] effects, superconductivity, and nonlinear phenomena. In particular, in the presence of supersymmetry for systems of gauge equations the ground state—the vacuum state—is degenerate, and the conditions for the occurrence of effects like the quantum Hall effect and nonstandard statistics are possible. In addition, supersymmetric quantum mechanics enables one to find exact solutions for a large class of problems, including many of the models obtained by the methods of the inverse scattering technique and Darboux transformations.[22–24]

In Refs. 5 and 25–28 there is discussed a generalization of supersymmetric quantum mechanics[29–33] to systems of gauge equations obtained in the adiabatic representation. The introduction of an additional scalar potential to preserve supersymmetry is considered. In the one-dimensional case, supersymmetry is completely determined by a matrix of scalar potentials, i.e., one can show here that the supersymmetry and gauge symmetry do not mix. However, in the multidimensional case the presence of supersymmetry may lead to the appearance of geometric phases (which, essentially, were written down by Aharanov and Casher in Ref. 34, which was devoted to a study of the degeneracy of the ground state of the Pauli Hamiltonian), the phases of Wilczek and Zee,[16] and various topological effects. Also possible is the occurrence of nonadiabatic geometric phases due to singularities of the vector potential at points of term crossing.[27,28,35]

The nondiagonal elements of the induced connection operator $A$ realize transitions between states of the parametric, so-called instantaneous Hamiltonian $H^f$ and generate nonadiabatic Aharanov–Anandan phases.[17] In a realistic formulation of the three-particle problem, the nondiagonal matrix elements of $A$ must be taken into account, since a correct solution of the problem is impossible without them. Moreover, near level crossing the adiabatic approximation is invalid. In the presence of crossing of two or even three levels, singularities of $A_{nm}$ arise and, as a result, phases induced by them. This makes it necessary to introduce geometric nonadiabatic matrices in the presence of singularities of $A$ in addition to the Aharanov–Anandan phases.[17,36] Berry phases[15] are obtained in the adiabatic limit when transitions between different states are ignored.

Definite success has now been achieved in the formulation of three-particle and many-particle scattering problems. The correct formulation and solution of scattering problems for systems of three charged particles are naturally obtained in the framework of the modified differential equations of Faddeev.[37] The exotic three-particle interaction—hyperspherically symmetric without two-particle potentials—reduces, essentially, to the ordinary radial formulations of the direct and inverse scattering problems.[39] Quite a large number of studies (in particular, those of Refs. 37–47, Refs. 6–10, and Refs. 48–59) have been devoted to the investigation of model three-particle scattering problems without rearrangement and realistic three-particle problems with allowance for scattering processes and redistribution and disintegration of the system into three fragments.

In Refs. 4 and 6–12 there was proposed a constructive solution of the direct and inverse scattering problems in a three-particle system, based on *a global adiabatic representation of the three-particle wave function* $\Psi$ *consisting of a sum of Faddeev components* $\Psi(\mathbf{X})=\Sigma_{\alpha=1}^{3}F_\alpha(\mathbf{X})$. By the use of an invariant adiabatic variable—the hyperradius $X=\sqrt{x_\alpha^2+y_\alpha^2}$—in three-particle scattering problems and representation of the Hamiltonian in the form $H=H^s(X)+H^f(X,\hat{\mathbf{X}})$ there is constructed a fiber space $\mathscr{H}$ with universal base $B=\mathbf{R}_+^1\ni X$ for all Faddeev components and trivial fiber $\mathscr{F}_X=L_2(\mathbf{S}_X^5(\hat{\mathbf{X}}))$ formed from elements of the basis. Correct boundary conditions corresponding to all possible processes of scattering with redistribution and disintegration follow from the Faddeev integral equations. The use of these boundary conditions, the hyperspherical adiabatic representation, and a generalized gauge transformation made it possible to carry out a correct reduction of the three-particle direct and inverse scattering problems to corresponding multichannel and parametric problems and to formulate them. The adiabatic approach is used to construct model three-particle potentials of Bargmann type and the corresponding exact solutions in closed analytic form.[4,11]

The class of exactly solvable quantum-mechanical problems is greatly extended by using the methods of inverse scattering problems[60–65] and Darboux–Crum–Kreĭn transformations.[66–68] In Refs. 69–76 there were proposed generalized Bargmann and Darboux–Crum–Kreĭn transformations that make it possible to construct in closed analytic form new series of potentials and corresponding solutions of the Schrödinger equation for variable values of the angular momentum $l$ and energy $E$ along arbitrary straight lines in the $(\lambda^2,E)$ plane $(\lambda=l+1/2)$. In the special case $l=$const, they go over into the ordinary expressions for the solutions and potentials of Bargmann type[77–81] in the Gel'fand–Levitan or Marchenko approaches with degenerate kernel of the generalized-shift operator and into transformations of the same type for $E=$const (see Refs. 39 and 82–85 and the references in them). In this sense, the approach gives a generalization of the exactly solvable models of the inverse problem with the advantage that it does not use the integral equations of the inverse problem and, accordingly, does not use explicitly the completeness of the set of eigenfunctions, which is needed for its derivation, and at the same time is a closed algebraic procedure. The resulting generalized Bargmann transformations are related to the generalized Darboux transformations in the same way that the corresponding ordinary transformations are related. The investigations for Schrödinger equations with variable values of the energy and orbital angular momentum made it possible to propose a technique of algebraic Bargmann and Darboux transformations for equations of more general form with a functional dependence on the right-hand side of the equation:

$$-d^2\phi(\gamma,r)/dr^2+V(R)\phi(\gamma,r)=\gamma^2h(r)\phi(\gamma,r) \qquad (1)$$

(Refs. 83 and 27), which find application in atomic physics, the theory of the propagation of electromagnetic waves, acoustics, geophysics, etc. For a definite choice of $h(r)$, the Bargmann and Darboux transformations for fixed values of $E$ and $l$ as well as for variable values are obtained as special cases of these generalized transformations.

## 2. ADIABATIC REPRESENTATION OF THE MULTIDIMENSIONAL SCATTERING PROBLEM

As is well known, the direct solution of a multidimensional inverse problem encounters many difficulties, mainly in connection with the fact that the problem is overdetermined. The fact is that the Gel'fand–Levitan and Marchenko methods are based on the existence of an integral kernel of generalized shift with the triangle property. Berezanskiĭ, in the finite-difference case,[87] and Faddeev, in the continuous case,[88] made significant advance in the search for the necessary triangle property in multidimensional problems. Then Faddeev (Ref. 88, 1971; Ref. 89) and after him independently Newton[90,93,94] investigated the two-dimensional inverse problem. It should be noted that the first attempt, already ten years before these studies, was made by Kay and Moses.[95,96] However, their method was designed for the reconstruction of nonlocal potentials and does not guarantee reconstruction of local ones. A finite-difference formulation of a two-dimensional inverse problem was realized in Refs. 39 and 97 on the basis of Berezanskiĭ's approach in polynomial problems of the recovery of an infinite Jacobi matrix.[87]

In recent studies, Beals and Coifman[98] and Novikov and Khenkin[99–101] investigated a different approach for two- and three-dimensional operators on a fixed level and energy surface. This was preceded by the inverse problem for fixed energy and variable values of the orbital angular momentum for a spherically symmetric decreasing potential, which was formulated in studies of Regge,[102] Newton,[103] Sabatier,[104] Loeffel,[105] Levitan,[85,106] and Lipperheide and Fiedeldey.[107,108]

It is of interest to look for and develop constructive methods of solving inverse problems by reducing them to problems of lower dimension. In this section, we consider a multidimensional inverse problem in the adiabatic approach, represented in the form of two inverse problems: one for a Schrödinger equation describing fast dynamics with parametric dependence on slow variables, and the other for the system of equations describing the slow motion. We give a method for constructing exactly solvable models for both cases and, thus, for the complete multidimensional problem.[4]

As we have already noted, the solution of complicated multidimensional problems is often based on a procedure of dimensional reduction of the space, $M=B\times\hat{M}$, by using an expansion of the wave function of the original Hamiltonian with respect to a complete set of known basis functions. In the general case, this corresponds to the introduction of Hilbert bundles $\mathscr{H}=\int_B\otimes\mathscr{F}_X d\mu(X)$, where $B$ is the base, $\mu(X)$ is a positive measure on it, and the fibers $\mathscr{F}_X$ form families of Hilbert spaces parametrized by points $X\in B$. In the traditional approach of the method of strong channel coupling, such a representation corresponds to a fixed fiber $\mathscr{F}_X$ formed from known basis functions, and the unknown coefficients, to the determination of which the problem reduces, are specified on the space $B$ of lower dimension than the original $M$. In contrast to this, the adiabatic representation, in which for the Hamiltonian $H$ one introduces the decomposition

$$H=h^s\otimes I+h^f, \qquad (2)$$

is formulated on a Hilbert bundle $\mathscr{H}$ with nonfixed fibers $\mathscr{F}_X$ formed from the eigenfunctions $\Phi_n(X,.)$ of the self-adjoint operators $h^f(X)$. Since the operators $h^f(X)$ act on the Hilbert fibers $\mathscr{F}_X$, it is convenient to call them fibers of the operator $H$: $H=\int_B\otimes h^f(X)dX$.[43] The operator $h^s\otimes I$ acts as $h^s$ with respect to the slow, $s$, variables $X$ and as the identity operator with respect to the fast, $f$, variables $\mathbf{Y}$. The total wave function of the system in such an approach is represented in the form of the expansion

$$\Psi(\mathbf{X})=\sum_n\!\!\!\!\!\!\int\ \Phi_n(X;.)\chi_n(X) \qquad (3)$$

with respect to the eigenstates $\Phi_n(X;.)$ of the self-adjoint Hamiltonian $h^f(X)$ of the fast motion for all fixed values of the slow variables $X$:

$$\begin{aligned} &h^f(X)\Phi_n(X;.)=\mathscr{E}_n(X)\Phi_n(X;.),\\ &h^f(X;.)=-\Delta_{\mathbf{Y}}+V^f(X,\mathbf{Y}). \end{aligned} \qquad (4)$$

The symbol $\Sigma\!\!\int$ in (3) denotes a summation over the states of the discrete spectrum $\mathscr{E}_n(X)\in\sigma_d(h^f(X))$ and an integration over the states of the continuous spectrum $\mathscr{E}_k(X)\in\sigma_c(h^f(X))$. If the functions are defined on a compact set of values $\mathbf{Y}\in\hat{M}$, they are all square-integrable, and the spectrum is purely discrete, as happens in the case of the hyperspherical parametrization of space, for which $\mathbf{Y}\in S^M$ is a set of angles. In the general case, since the scattering states $\Phi_k(X;.)\equiv\Phi(k,X;.)$ form together with the states of the discrete spectrum a complete set, they must be taken into account in the expansion (3), although they do not belong to $L_2$. Thus, let $B\ni X$ be a smooth manifold of dimension $N$ and the period denote the fast variables $\mathbf{Y}\in\hat{M}$, which belong to the space $\hat{M}$ of dimension $M$. Depending on the particular formulation of the problem, we shall use either a compact or a noncompact base manifold.

Substitution of the expansion (3) in the original Schrödinger equation

$$H\Psi(\mathbf{X})=E\Psi(\mathbf{X}) \tag{5}$$

and averaging over the fast variables $\mathbf{Y}$ of the moving frame $\Phi_n(X,\mathbf{Y})$ leads to a multichannel scattering problem for the coefficients $\chi$:

$$[-(\nabla\otimes I-iA(X))^2+U(X)\otimes I-P^2]\chi(X)=0$$

$$(P=\mathrm{diag}(p_n)). \tag{6}$$

Here, $A(X)$ is the effective vector potential generated by the basis functions:

$$A_{nm}(X)=i\langle\Phi_n|\nabla_X|\Phi_m\rangle. \tag{7}$$

The effective scalar potential

$$U(X)=U^f(X)+U^s(X)$$

consists of a diagonal potential matrix $U^f(X)=\mathrm{diag}\{\mathscr{E}_m(X)\}$,

$$\sum_n U^f_{nm}(X)=\sum_n\langle\Phi_n(X;.)|h^f(X)|\Phi_m(X;.)\rangle=\mathscr{E}_m(X), \tag{8}$$

the elements of which are identical to the energy levels $\mathscr{E}_m(X)$ of the Hamiltonian $h^f(X)$ (4) and a certain additional potential matrix $U^s(X)$, which is present only in the system of equations (6) and depends on the slow variables:

$$U^s_{nm}(X)=\langle\Phi_n(X;.)|V^s(X,.)|\Phi_m(X;.)\rangle. \tag{9}$$

In general, the system of equations (6) is an integrodifferential system, since besides the summation over the states of the discrete spectrum of the operator $h^f(X)$ it is necessary to integrate over the states of the continuous spectrum (scattering spectrum) of the operator $h^f(X)$. Thus, if $n$ and $m$ label the discrete spectrum of the operator $h^f(X)$, the corresponding part $\|A_{nm}(X)\|$ is a matrix, and if $n$ and $m$ label the continuous spectrum of the operator $h^f(X)$, then the corresponding part $\|A_{nm}(X)\|$ is an integral operator with kernel $A_{nm}(X)$, i.e., in general a generalized function, since the continuum functions $\{\Phi_n(X,.)\}$ do not belong to $L_2$. The problem of solving integrodifferential equations is a common one for the method of strong channel coupling. In actual problems, one often makes a restriction of the expansion to only a finite discrete set of states of the spectrum, and this is more justified, the more rapid is the convergence of the expansion with respect to some set of basis functions. The contribution of the terms omitted in (28) can be taken into account effectively by using a projection technique of Feshbach type (see, for example, Ref. 43, in which allowance is made in the framework of the adiabatic representation for the contributions of the channels closed with respect to the energy to the scattering data).

The expansion (3) is valid if in each fiber $\mathscr{F}_X$ the basis functions form a complete orthonormal set $\{\Phi_n(X,.)\}$ of eigenfunctions $h^f(X)$, as follows from the requirement that the operators $h^f(X)$ be self-adjoint for every $X\in B$:

$$\langle n|m\rangle=\int\Phi^\dagger_n(X,Y)\Phi_m(X,Y)dY=\delta_{nm};$$

$$\langle q|q'\rangle=\delta(q-q') \tag{10}$$

$$\sum_n\Phi_n(X,Y)\Phi^\dagger_n(X,Y')=\delta(Y-Y').$$

Note that the representation (3) of the total wave function $\Psi$ must be invariant under the unitary transformation

$$\Phi_m(\mathbf{X};\mathbf{Y})=\sum_n\Phi'_n(\mathbf{X};\mathbf{Y})\mathscr{U}_{nm}(\mathbf{X});$$

$$\chi_m(\mathbf{X})=\sum_n\mathscr{U}^*_{nm}(\mathbf{X})\chi'_n(\mathbf{X}).$$

Using the relation between the basis functions $\{\Phi\}$ and $\{\Phi'\}$, one can show[115] that the induced connection operator $A$ really is a gauge field or an affine connection, i.e., that on a change of the basis in the Hilbert space $\mathscr{F}$ it transforms in accordance with the rule

$$A'_\mu=\mathscr{U}A\mathscr{U}^{-1}-i\mathscr{U}^{-1}\partial_\mu\mathscr{U}.$$

In accordance with this, the extended derivative

$$D_\mu=\partial_\mu\otimes I-iA_\mu(X)$$

that occurs in the effective equation (6) is called a covariant derivative. Using what was said above, we define at a certain fixed point $X=X_0$ the frame

$$|e(Y)\rangle\equiv|\Phi(X_0;Y)\rangle. \tag{11}$$

The moving frame $|\Phi(X;.)\rangle$ is related to the fixed one $|e(.)\rangle$ by means of the unitary bilocal operator $\mathscr{U}(X)\equiv\mathscr{U}(X,X_0)$, $\mathscr{U}^+(X)=\mathscr{U}^{-1}(X)$:

$$|\Phi(X;.)\rangle=|e\rangle\mathscr{U}(X,X_0),\quad \mathscr{U}(X;X_0)=\langle e|\Phi(X;.)\rangle, \tag{12}$$

which realizes parallel transport of the frame from $X_0$ to $X$. From the definition (7) of the operator $A$ we obtain, with allowance for (11) and (10),

$$A_\nu=i\mathscr{U}^{-1}\partial_\nu\mathscr{U}.$$

Then the covariant derivative can be expressed in terms of $\mathscr{U}$:

$$D_\nu=\partial_\nu\otimes I+\mathscr{U}^{-1}\partial_\nu\mathscr{U}$$

and

$$\mathscr{U}(X,X_0)=P\exp i\int_X^{X_0}A(X')dX'. \tag{13}$$

Since as a result of the gauge transformation the effective scalar and vector potential matrices in (6) take the form

$$U'(X)=\mathscr{U}(X)U(X)\mathscr{U}^{-1}(X), \tag{14}$$

$$A'(X)=\mathscr{U}A\mathscr{U}^{-1}-i\mathscr{U}^{-1}\partial_X\mathscr{U}, \tag{15}$$

it is easy to show that under the transformation (13) the matrix of the vector potential vanishes in the absence of singularities of $A(X)$, and the matrix of the scalar potential can be expressed in the representation of the fixed basis $|e\rangle$. Then the system of equations (6) reduces to an ordinary multichannel system of equations with potential coupling

$$\left\{-\frac{d^2}{dX^2}+U'(X)-P^2\right\}\chi'(X,P)=0 \tag{16}$$

for new coefficients $\chi'$, which are related to the old ones $\chi$ by

$$\chi'(X,P)=\mathscr{U}(X)\chi(X,P). \tag{17}$$

As is well known from vector analysis, by making a gauge transformation on the vector potential one can achieve its annihilation in the cases when curl $A=0$. For non-Abelian gauge fields, with which we shall deal in the adiabatic representation, this condition corresponds to vanishing of the matrix tensor

$$R_{\mu\nu}=\partial_\mu A^\nu-\partial_\nu A^\mu-i[A^\nu,A^\mu]$$

(in geometrical language, $R_{\mu\nu}$ is the curvature). Nontrivial topological effects (for example, the Hall effect) occur precisely when $R_{\mu\nu}\neq 0$. If the vector potential is singular at certain points of the coordinate space $X\in B$, for example, at points of crossing of terms, then the vanishing of $R_{\mu\nu}$ is no longer sufficient to eliminate the vector potential at all $X$. Here, we shall assume fulfillment of the condition $R_{\mu\nu}=0$ and the absence of singularities of **A**.

We can now apply the standard methods of the multichannel inverse problem for the system of equations (16) if we know its corresponding $S$ matrix $S'(P)$ and information on the states of the discrete spectrum, namely, their positions $\{E_\lambda\}$ and normalizations $\{M'_\lambda\}$. Because of the unitary freedom in the gauge of the radial functions, it can be shown that

$$M'_\lambda=M_\lambda,\quad \hat{S}'(P)=\hat{S}(P). \tag{18}$$

Indeed,[4] we substitute in the definition of the normalization matrices $M'_\lambda$ the connecting relation (17) and use the unitarity of the matrices $\mathscr{U}(X)$. Then

$$M'_\lambda=\left[\int_0^\infty \tilde{F}'(i\kappa_\lambda,X)F'(i\kappa_\lambda,X)dX\right]^{-1}$$

$$=\left[\int_0^\infty \tilde{F}(i\kappa_\lambda,X)\mathscr{U}^{-1}(X)\mathscr{U}(X)F'(i\kappa_\lambda,X)dX\right]^{-1}$$

$$=M_\lambda\quad ((i\kappa_\lambda)^2=E_\lambda<0). \tag{19}$$

An analogous derivation of the second of the relations (18) was proposed in Ref. 10. As is well known, a symmetric matrix $\hat{S}'$ that is unitary with respect to the open channels and corresponds to the system (16) is determined as follows:

$$\hat{S}'(P)=P^{-1/2}\mathscr{F}'_-(P)(\mathscr{F}'_+(P))P^{1/2}. \tag{20}$$

Using the relation (17) in the standard definition of the Jost matrix functions $\mathscr{F}_\pm(P)$, we obtain

$$\mathscr{F}'_\pm(P)$$

$$=W\{F'_\pm(X,P),\Phi'(X,P)\}=W\{F_\pm(X,P),\Phi(X,P)\}$$

$$+2\tilde{F}^*_\pm(X,P)A(X)\Phi(X,P)$$

$$=W_d\{F_\pm(X,P),\Phi(X,P)\}\equiv\mathscr{F}_\pm(P). \tag{21}$$

Here, $\Phi'$ and $\Phi$ are the matrices of the regular solutions of the systems of equations (16) and (6), respectively. The tilde is used to denote the transpose. However, the $\hat{S}$ matrix corresponding to the system (6) is determined by such matrix Jost functions:

$$\hat{S}(P)=P^{-1/2}\mathscr{F}_-(P)(\mathscr{F}_+(P))^{-1}P^{1/2}. \tag{22}$$

The upshot is that, taking into account (21), we obtain (18). So far, we have assumed that there is no quasicrossing of the levels $\mathscr{E}_n(X)$ responsible for violating the unitarity of $\mathscr{U}$ at these points, no resulting singular terms in (16), and no nontrivial geometric phase in the scattering data.[26,27] Berry[15] demonstrated the existence of monopole fields in simple dynamical systems that arise naturally in the framework of gauge theory.[16] If we wish to estimate the linear integral (13) around a closed contour $C$, bringing $X$ back to $X_0$ after the circuit, it is better, using Stokes's theorem, to replace it by a surface integral:

$$\gamma=\oint_C A(X)dX=\int\int_S BdS,$$

where $B=\nabla\times A$ (in our case $B=R_{\mu\nu}$). Since curl grad$=0$, the result does not depend on the gauge transformation $A\rightarrow A-\nabla\chi$. Following the same logic, the surface integral can be replaced in accordance with Gauss's theorem by a volume integral, for which we can expect a zero result, since div curl$=0$. However, in the case of intersection of the potential curves at a certain point of the $R$ space, the vector potential $A(X)$ is singular and the result of integrating around a closed contour is not zero. There arises a geometric phase, which was encountered long ago in atomic physics.[19] However, it was explained in the celebrated study of Berry[15] (see also Refs. 35 and 114). This problem will be discussed later (Sec. 9). Here we consider a simple example of the occurrence of a phase for free motion in an $N$-dimensional space.

### 2.1. Geometric phase of free solutions in *N*-dimensional space

Free motion in $\mathbf{R}^N\setminus\{0\}$ is described in spherical coordinates by the equation

$$\frac{N-1}{X}\partial_X\Psi-\partial_X^2\Psi-X^{-2}\nabla_{\hat{X}}\Psi=E\Psi$$

$$(X\in\mathbf{R}_+^1,\hat{\mathbf{X}}\in\mathbf{S}^{N-1}). \tag{23}$$

We rewrite this equation in terms of the extended derivative $D(X)=\partial_X-iA(X)$:

$$[(-\partial_X+\nu X^{-1})^2+\nu(\nu-1)X^{-2}-X^{-2}\nabla_{\hat{X}}]\Psi=E\Psi. \tag{24}$$

Here, $\nu=(N-1)/2$. Since

$$AdX=i\nu X^{-1}dX=i\mathscr{U}^{-1}d\mathscr{U},$$

we obtain $\mathscr{U}=X^\nu$. In accordance with (17), we introduce the new functions $\Psi'$:

$$\Psi=\mathscr{U}^{-1}\Psi'=X^{-\nu}\Psi'.$$

After substitution of such a $\Psi$ in (24), we obtain an equation for $\Psi'$ without a first derivative:

$$-\partial_X^2\Psi'+\nu(\nu-1)X^{-2}\Psi'-X^{-2}\nabla_{\hat{X}}\Psi'=E\Psi'. \tag{25}$$

The nonvanishing centrifugal barrier $\nu(\nu-1)X^{-2}$ is associated with the defect of the embedding of the sphere $\mathbf{S}^{N-1}$ in $\mathbf{R}^N\setminus\{0\}$. Only for one-dimensional, $\nu=0$, and three-dimensional, $\nu=1$, spaces is this absent. The presence of the barrier $\nu(\nu-1)X^{-2}$ leads to the appearance of a topological phase $\delta$, which is readily found by calculating the integral by means of the rules of contour integration:

$$\delta=\mathrm{Im}\int_C A(X)dX=\mathrm{Im}\int_C\frac{\nu dX}{X}=\pi\nu. \tag{26}$$

We now consider an $N$-dimensional scattering problem with a spherically nonsymmetric perturbing interaction $V(\mathbf{X})$. Let the radius $X\in\mathbf{R}_+^1$ of the "sphere" $\mathbf{S}^{N-1}$ be the slow variable, and the angles $\hat{\mathbf{X}}\in\mathbf{S}^{N-1}$ be the fast variables. Then the parametric fast equation (4) can be rewritten in the form

$$h^f(X,\hat{\mathbf{X}})\Phi_n(X;\hat{\mathbf{X}})=[-X^{-2}\nabla_{\hat{\mathbf{X}}}+V^f(X,\hat{\mathbf{X}})]\Phi_n(X;\hat{\mathbf{X}})$$

$$=\mathscr{E}_n(X)\Phi_n(X;\hat{\mathbf{X}}), \tag{27}$$

and the system of slow equations (6) will take the form

$$[-(\partial_X-iA)^2+\nu(\nu-1)X^{-2}+U(X)-P^2]\chi(X,P)=0, \tag{28}$$

$$U(X)=U^s(X)+\langle\Phi|h^f(X,\hat{\mathbf{X}})|\Phi\rangle. \tag{29}$$

The matrix elements of the effective vector and scalar potentials are obtained by averaging over the angles of the moving frame $|\Phi(X;\hat{\mathbf{X}})\rangle$, which satisfies (27).

## 3. MULTICHANNEL INVERSE PROBLEM

In the generalized approach of Marchenko,[109–111] with the reference potential $\mathring{V}(X)\neq0$, the basic matrix equations of the inverse problem that correspond to the system of equations (16) are

$$K(X,X')+Q(X,X')+\int_X^\infty K(X,t)Q(t,X')dt=0, \tag{30}$$

$$U'(X)=\mathring{U}'(X)-2\frac{d}{dX}K(X,X), \tag{31}$$

$$F'(X,P)=\mathring{F}'(X,P)+\int_X^\infty K(X,X')\mathring{F}'(X',P)dX'. \tag{32}$$

The matrices of the Jost solutions $F'(X,P)$, $\mathring{F}'(X,P)$ are related to the Jost solutions $F(X,P)$, $\mathring{F}(X,P)$ of the system of equations (6) or (28) in accordance with (17). The system of Marchenko equations (30) can be solved for the matrix kernel of generalized shift $K(X,t)$ for the known kernel $Q(X,t)$, which is determined by the scattering data $\hat{S}'(P)$, $\{M'_\lambda\}$, $\{E_\lambda\}$ and $\mathring{\hat{S}}'(P)$, $\{\mathring{M}'_\lambda\}$, $\{\mathring{E}_\lambda\}$ corresponding to the system of equations (16) with potential matrices $U'(X)$ and $\mathring{U}'(X)$:

$$Q(X,X')=\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathring{F}'(X,P)[\mathring{\hat{S}}'(P)-\hat{S}'(P)]\widetilde{\mathring{F}}'$$

$$\times(X',P)dP+\sum_\lambda^N\mathring{F}'(X,i\kappa_\lambda)M'_\lambda\widetilde{\mathring{F}}'(X',i\kappa_\lambda)$$

$$-\sum_\lambda^{\mathring{N}}\mathring{F}'(X,i\mathring{\kappa}_\lambda)\mathring{M}'_\lambda\widetilde{\mathring{F}}'(X',i\mathring{\kappa}_\lambda). \tag{33}$$

The Jost solutions $\mathring{F}'(X,i\kappa_\lambda)$ and $\mathring{F}'(X,i\mathring{\kappa}_\lambda)$ of the system (16) with the matrix $\mathring{U}'(X)$ must be taken at the energies $E_\lambda$ and $\mathring{E}_\lambda$ of the bound states of both problems with $U'(X)$ and $\mathring{U}'(X)$, respectively.

### 3.1. Reconstruction of the interaction potential $V^s(\mathbf{X})$ characterizing the slow dynamics

We first investigate the simplest situation in which the potential $V^f(\mathbf{X})$ is known in advance. In the three-particle problem, this could be an effective potential $V^f(\mathbf{X})=\Sigma_\alpha V_\alpha(\mathbf{X})$ equal to a sum of two-particle potentials. It is necessary to determine the additional potential $V^s(\mathbf{X})$. (In the considered example, $V^s(\mathbf{X})=V_{123}(\mathbf{X})$ is the potential of a three-particle interaction.) We shall assume that the amplitudes $f^f(\mathbf{P},\hat{\mathbf{X}})\equiv\mathring{f}(\mathbf{P},\hat{\mathbf{X}})$ and $f(\mathbf{P},\hat{\mathbf{X}})$ are known. For each fixed value of the slow variable $X$ we determine from the solution of the direct eigenvalue problem for Eq. (4) the frame functions $\Phi_n(X,\hat{\mathbf{X}})$ and the terms $\mathscr{E}_n(X)$, which depend parametrically on $X$. Besides the parametric Sturm–Liouville problem, we can consider for Eq. (4) scattering problems or problems with periodic boundary conditions. From the known basis functions $\Phi(X;.)$, we find the matrix elements of the connection operator $A(X)$ (7), and then, using Eqs. (13) or (11), the bilocal transport operator $\mathscr{U}(X)$. The unitary operator $\mathscr{U}(X)$ enables us to go over from the system of equations (28) to the system (16) with as yet an unknown potential $V(X;.)\equiv V^s(X;.)$ in the matrix elements of the fixed frame $|e\rangle$.

The physical asymptotic behaviors of the solutions of the system of equations (28),

$$\chi(X,P) \underset{X\to\infty}{\to} -(2i)^{-1}\{\exp(-i(X,P-\pi\gamma/2))\otimes \mathbf{1} -\exp(i(XP-\pi\gamma/2))P^{-1/2}\hat{S}(P)R^{1/2}\}, \quad (34)$$

are determined by the known asymptotic behavior of the total wave function:

$$\Psi(\mathbf{X},\mathbf{P}) \underset{X\to\infty}{\to} (2\pi)^{-3/2-\gamma}[\exp i(\mathbf{X}\cdot\mathbf{P}) -X^{-\gamma-1}\exp i(XP-\pi\gamma/2)f(\hat{\mathbf{X}},\mathbf{P})]. \quad (35)$$

We have introduced $\gamma=\nu-1$, using the traditional form of expression, in which one separates the part of the phase without the defect of the embedding of the sphere $\mathbf{S}^2\in\mathbf{R}^3$; $\hat{S}(P)$ is a symmetric $S$ matrix,

$$\hat{S}(P)=P^{1/2}\bar{S}(P)P^{-1/2}, \quad \hat{S}=\tilde{\hat{S}}, \quad (36)$$

unitary on the open channels.

Since we assume that the amplitudes $f(\hat{\mathbf{X}},\mathbf{P})$ and $\mathring{f}(\hat{\mathbf{X}},\mathbf{P})\equiv f^f(\hat{\mathbf{X}},\mathbf{P})$ are given, we find the corresponding matrices $\hat{S}'(P)$ and $\mathring{\hat{S}}'(P)$, using the relations (18) and the relationship between the total multidimensional amplitude and the partial amplitudes:[4,7]

$$f(\hat{\mathbf{X}},\mathbf{P})=\frac{4\pi}{2i}\sum_{nm}\Phi_n(\hat{\mathbf{X}},\infty)\hat{f}_{nm}(P)\Phi_m^\dagger(-\mathbf{P},\infty),$$
$$\hat{f}_{nm}(P)=(\hat{S}(P)-\hat{I}\cdot\mathbf{1})_{nm}P_m^{-1-\gamma}, \quad (37)$$

where $\hat{I}$ is the operator of complete inversion in $\mathbf{R}^N\setminus 0$;

$$\hat{S}(P)=\hat{I}S(P), \quad (38)$$

and $S$ is the ordinary scattering operator.[112,113]

It has been assumed here that the system of equations (28) is finite. If a complete set of states is included in the expansion (3), it is exact. Usually, the expansion (3), which in the general case contains states $\Phi(X,.)$ of the discrete and continuous spectra of the Hamiltonian $h^f$, is restricted to a finite (incomplete) set of $N$ states.

The contribution of the omitted terms in (6) can be taken into account by using a projective technique of Feshbach type.[43] This leads to the appearance of additional potential terms in a system consisting of a finite set of "slow" equations. A system of a restricted number of equations was investigated in Ref. 115. The contribution of the remainder terms was ignored, assuming that they are small.

The basic generalized equations of the multichannel inverse Gel'fand–Levitan–Marchenko problem (30)–(32) now make it possible to determine the potential matrix $\langle e|V^s(X,\hat{\mathbf{X}})|e\rangle$ and the matrix solutions corresponding to it.

Returning to the representation in the basis $\Phi(X,.)$, we obtain the following relations in Marchenko's approach:

$$U^s(X)=U(X)-U^f(X) =-2\mathcal{U}^{-1}(X)\frac{d}{dX}K(X,X)\mathcal{U}(X), \quad (39)$$

$$F_\pm(X,P)=\mathcal{U}^{-1}(X)F'_\pm(X,P) =\mathcal{U}^{-1}(X)\Big[\mathring{F}'_\pm(X,P) =\int_X^\infty K(X,X')\mathring{F}'_\pm(X',P)dX'\Big]. \quad (40)$$

Here, $\mathcal{U}(X)\equiv\mathcal{U}(X,\mathring{X})$ (it is convenient to choose $\mathring{X}\to\infty$), $\mathring{F}'_\pm(X,P)$ are matrix Jost solutions of the system of equations (16) with potential matrix $\mathring{U}'(X)=U^{f'}(X)$.

The physical solutions of the system (6) or (28) are obtained as a linear combination of the Jost solutions:

$$\chi^{ph}(X,P)=-(2i)^{-1}\{F^-(X,P) -F^+(X,P)P^{-1/2}\hat{S}(P)P^{1/2}\}. \quad (41)$$

The matrix Gel'fand–Levitan equations differ from the basic Marchenko equations only by the sign in (31) and (39) and in the limits of integration in (40), (30), and (32):

$$U^s(X)=U(X)-U^f(X) =2\mathcal{U}^{-1}(X)\frac{d}{dX}K^{\mathrm{GL}}(X,X)\mathcal{U}(X), \quad (42)$$

$$\Phi(X,P)=\mathcal{U}^{-1}(X)\Phi'(X,P) =\mathcal{U}^{-1}(X)\Big[\mathring{\Phi}'(X,P) +\int_0^X K^{\mathrm{GL}}(X,X')\mathring{\Phi}'(X',P)dX'\Big], \quad (43)$$

where $K^{\mathrm{GL}}(X,X')$ is the matrix integral kernel of the Gel'fand–Levitan equation,

$$K^{\mathrm{GL}}(X,X')+Q^{\mathrm{GL}}(X,X') +\int_0^X K^{\mathrm{GL}}(X,Y)Q^{\mathrm{GL}}(Y,X')dY=0, \quad (44)$$

and it is determined from the given $Q^{\mathrm{GL}}(X,X')$:

$$Q^{\mathrm{GL}}(X,X')=\int_{-\infty}^\infty \mathring{\Phi}'(X,P)d(\rho'(P)-\mathring{\rho}'(P))\tilde{\mathring{\Phi}}'(X',P). \quad (45)$$

Here, $\rho'(P)$ and $\mathring{\rho}'(P)$ are the spectral matrices corresponding to the system (16):

$$\frac{d\rho'(P)}{dE}=\frac{\rho}{\pi}|\mathcal{F}'(P)|^{-2}, \quad E\geqslant 0,$$
$$\frac{d\rho'(P)}{dE}=\sum_\lambda\delta(E-E_\lambda)O'_\lambda, \quad E<0. \quad (46)$$

The normalization matrix $O'_\lambda$ is determined by the relation

$$O'_\lambda=\Big[\int_0^\infty\big|\Phi'(X,P)\big|^2dX\Big]^{-1}, \quad (47)$$

and the matrix regular solutions $\Phi(X,P)$ of the system of equations (28) by the boundary conditions

$$\lim_{X\to 0}\Phi(X,P)X^{-(\mathscr{K}+\nu)}=1, \tag{48}$$

where $\mathscr{K}$ is the hypermoment.

Because $\mathscr{F}'(P)=\mathscr{F}(P)$ (21), we obtain, as in the case of (18), $\rho'(P)=\rho(P)$ and $O'_\lambda=O_\lambda$.

## 4. MULTICHANNEL EXACTLY SOLVABLE MODELS

For potentials of Bargmann type, the kernel $Q(X,X')$ can be represented as a sum of factorized terms, by virtue of which the systems of integral equations (30)–(32) reduce to algebraic equations.

Note that in the construction of exactly solvable models by the methods of the inverse problem it is necessary to satisfy requirements on the scattering data, under which they correspond to the corresponding boundary-value problem for the Schrödinger equation or system of Schrödinger equations, in the given case (28) and (34), with a local and bounded potential matrix:[64]

$$\int_0^\infty X|U(X)|dX<\infty,\qquad \int_0^\infty |U(X)|dX<\infty. \tag{49}$$

On the basis of the results of Ref. 117, it is easy to write down an algebraic scheme for solving a multichannel inverse problem in the adiabatic representation.

We consider a fairly simple situation in which the required potential $U^s(X)$ adds a few bound states, leaving the scattering data or the spectral characteristics of the problem (28) with the potential matrix $U^f(X)\equiv\mathring{U}(X)$ unchanged. This can correspond to a form of the inverse problem on the complete axis in the Marchenko approaches with $\hat{S}(P)=\mathring{\hat{S}}(P)$:

$$Q^M(X,X')=\sum_\lambda^N \mathring{F}'(X,i\kappa_\lambda)M'_\lambda\tilde{\mathring{F}}'(X',i\kappa_\lambda)$$
$$=\mathring{F}'^T(X)M\mathring{F}'(X') \tag{50}$$

or to the radial problem in the Gel'fand–Levitan approach with equal spectral matrices $\rho(P)=\mathring{\rho}(P)$ for $E>0$:

$$Q^{GL}(X,X')=\sum_\lambda^N \mathring{\Phi}'(X,i\kappa_\lambda)O'_\lambda\tilde{\mathring{\Phi}}'(X',i\kappa_\lambda)$$
$$=\mathring{\Phi}'^T(X)O\mathring{\Phi}'(X'). \tag{51}$$

In this abbreviated notation, the kernel $Q(X,X')$ is a "supermatrix" with respect to both the indices $\lambda$ of the bound states and the indices $\alpha$ of the channels. We first used such a notation in Ref. 118, which was devoted to analytic solution of an inverse problem in the $R$-matrix formulation.

In the relation (50), the column vector $\mathring{F}(X)$ and row vector $\mathring{F}^T(X)$ are formed from $N$ elements $\mathring{F}(X,i\kappa_\lambda)$, each of which is an $m\times m$ matrix with respect to the channel number $\alpha$; $M$ is an $N\times N$ diagonal matrix of normalized coefficients, the elements of which are also $m\times m$ matrices. The relation (51) for $Q^{GL}(X,X')$ is expressed similarly. Here, $\mathring{\Phi}(X)$ is a column vector and $\mathring{\Phi}^T(X)$ is a row with respect to the indices $\lambda$ of the bound states, their individual elements being $m\times m$ matrices of regular solutions with respect to the channel states. The transposed matrices with respect to the channel numbers $\alpha$ are denoted by $\tilde{\mathring{\Phi}}$ and $\tilde{\mathring{F}}$, in contrast to $\mathring{\Phi}^T(X)$ and $\mathring{F}^T(X)$ with respect to the indices $\lambda$.

### 4.1. Factorization of normalization matrices

We study a question relating to the properties of the normalization matrices of the bound states. Since the system (16) is an ordinary system with a Hermitian-conjugate potential matrix, its physical matrix solutions satisfy the completeness relation

$$(2/\pi)\int_0^\infty \chi'(X,P)\chi'^\dagger(X,P)dP+\sum_\lambda \chi'(i\kappa_\lambda,X)\tilde{\chi}'(i\kappa_\lambda,X')$$
$$=\mathbf{1}\otimes\delta(X-X'). \tag{52}$$

By virtue of the reduction of the system (6) to (16), the formulation of the inverse problem has been made possible.

The solution vectors corresponding to bound states can be orthonormalized:

$$\int_0^\infty \tilde{\chi}'(i\kappa_\lambda,X)\chi'(i\kappa_{\lambda'},X)dX=\delta_{\lambda\lambda'}. \tag{53}$$

Written out in full, this last relation takes the form

$$\sum_i^m\int_0^\infty \chi'_j(i\kappa_\lambda,X)\chi'_j(i\kappa_{\lambda'},X)dX=\delta_{\lambda\lambda'}.$$

Each element of the vector $|\chi'_j(i\kappa_\lambda,X)\rangle$ with respect to the channel indices is obtained as a linear combination of elements of the matrix of Jost solutions:

$$|\chi'(i\kappa_\lambda,X)\rangle=F'(i\kappa_\lambda,X)|\Gamma_\lambda\rangle \tag{54}$$

or

$$|\chi'_i(i\kappa_\lambda,X)\rangle=\sum_j F'_{ij}(i\kappa_\lambda,X)\gamma_j^{(\lambda)}.$$

Then

$$\langle\chi'(i\kappa_\lambda,X)|\chi'(i\kappa_\lambda,X)\rangle$$
$$=\left\langle\Gamma_\lambda\right|\int_0^\infty \tilde{F}'(i\kappa_\lambda,X)F'(i\kappa_\lambda,X)dX\left|\Gamma_\lambda\right\rangle=1.$$

Hence

$$\left[\int_0^\infty \tilde{F}'(i\kappa_\lambda,X)F'(i\kappa_\lambda,X)dX\right]^{-1}=|\Gamma_\lambda\rangle\langle\Gamma_\lambda|=M'_\lambda, \tag{55}$$

i.e., the normalized matrices of the states of the discrete spectrum can be represented in a form factorized with respect to the channel indices.

The factorization of the normalization matrices of the regular solutions is analogous:

$$O'_\lambda=|C_\lambda\rangle\langle C_\lambda|. \tag{56}$$

The use of (55) and (56) in (50) and (51) makes it possible to represent the matrix kernel $Q(X,Y)$ as a sum of $N$ terms, factorized with respect to both the coordinates and the channel indices, and as a result of this it is possible to give simple analytic expressions for potential matrices of Bargmann type and the solutions corresponding to them.[117,119]

### 4.2. Algebraic relations of the inverse problem

Bearing in mind the factorization of the normalization matrices in Marchenko's approach, we can write the kernel $Q(X,X')$ in the form

$$Q^M(X,X') = \sum_{\lambda}^{N} \mathring{F}'(X,i\kappa_\lambda)|\Gamma_\lambda\rangle \left\langle \Gamma_\lambda \right| \tilde{\mathring{F}}'(X',i\kappa_\lambda)$$
$$= \sum_{\lambda}^{N} \left|\mathring{\chi}'(X,i\kappa_\lambda)\right\rangle \langle \mathring{\chi}'(X',i\kappa_\lambda)|$$
$$= \mathring{\chi}'^T(X)\mathring{\chi}'(X'). \quad (57)$$

We have here used the notation

$$|\mathring{\chi}'(X,i\kappa_\lambda)\rangle = \mathring{F}'(X,i\kappa_\lambda)|\Gamma_\lambda\rangle \quad (58)$$

or, in component form,

$$\mathring{\chi}'_i(X,i\kappa_\lambda) = \sum_{i}^{m} \mathring{F}'_{ij}(X,i\kappa_\lambda)\gamma_j^\lambda.$$

We can write in the same factorized form as (57) the kernel $Q^{GL}$ of the Gel'fand–Levitan equations; here, the vector $|\mathring{\chi}'\rangle$ (58) is replaced by $|\mathring{\chi}'^{GL}\rangle$, which is combined from matrix elements of the regular solutions $\mathring{\Phi}'$:

$$|\mathring{\chi}'^{GL}(X,i\kappa_\lambda)\rangle = \mathring{\Phi}'(X,i\kappa_\lambda)|C_\lambda\rangle. \quad (59)$$

We give algebraic relations for the matrix kernel of generalized shift $K(X,X')$ of the potential matrices and solutions,[117] which are now obtained from the basic equations of the inverse problem (30)–(32), which is no more complicated than in the single-channel case. The orthogonalization matrix $K(X,X')$, like $Q(X,X')$, can be represented as a sum of factorized terms:

$$K^M(X,X') = -\sum_{\lambda}^{N} |\chi'(X,i\kappa_\lambda)\rangle\langle\mathring{\chi}'(X',i\kappa_\lambda)|$$
$$= -\chi'^T(X)\mathring{\chi}'(X'), \quad (60)$$

where each element of the vector of solutions (54) is found after the substitution of (57) and (60) in (30):

$$|\chi'(X,i\kappa_\lambda)\rangle = \sum_{\nu}^{N} |\mathring{\chi}'(X,i\kappa_\nu)\rangle P_{\nu\lambda}^{-1}(X). \quad (61)$$

At the same time, the matrix elements $P_{\nu\lambda}(X)$ do not depend on the channel indices:

$$P_{\nu\lambda}(X) = \delta_{\nu\lambda} + \sum_{j}^{m} \int_X^\infty \mathring{\chi}'_j(X',i\kappa_\nu)\mathring{\chi}'_j(X',i\kappa_\lambda)dX'. \quad (62)$$

Then

$$K(X,X') = -\mathring{\chi}'^T(X)P^{-1}(X)\mathring{\chi}'(X'), \quad (63)$$

or, in matrix form,

$$K_{ij}(X,X') = -\sum_{\nu\lambda} \mathring{\chi}'_i(X,i\kappa_\nu)P_{\nu\lambda}^{-1}(X)\mathring{\chi}'_j(X',i\kappa_\lambda).$$

As follows from the relations (31) and (32), the Bargmann potential matrix and the Jost solutions for all momenta $P$ can be expressed in the form

$$U'(X) = \mathring{U}'(X) = 2\frac{d}{dX}\mathring{\chi}'^T(X)P^{-1}(X)\mathring{\chi}'(X),$$
$$(\mathring{U}'(X) \equiv U^{f'}(X)), \quad (64)$$
$$F'_\pm(X,P) = \mathring{F}'_\pm(X,P) - \mathring{\chi}'^T(X)P^{-1}(X)$$
$$\times \int_X^\infty \mathring{\chi}'(X')\mathring{F}'_\pm(X',P)dX'. \quad (65)$$

Note that if we do not take into account the separability of the normalization matrices $M$ and $C$ with respect to the channel indices (55), (56) and, therefore, of the kernels $Q^M$ and $Q^{GL}$ (57) not only with respect to the coordinates but also with respect to the channels, then in the calculation of each of the matrix elements of $K$, $U$, $F_\pm$ it is necessary to invert $Nm \times Nm$ matrices, as was shown in Refs. 119 and 117. Thus, although a normalization matrix did appear in factorized form in Ref. 120, it was not used to simplify the expressions of the multichannel inverse problem.

The derivation of the relations (64) and (65) for a level shift realized in problems with different potential matrices $U(X)$ and $\mathring{U}(X)$ is analogous. In this case, the relations (57) for $Q^M$ and $Q^{GL}$ are written down with subtraction of one further sum of factorized terms, which are determined by the spectral characteristics $\{\mathring{\gamma}_\lambda, \mathring{\kappa}_\lambda\}$ of the original potential matrix $\mathring{U}'(X)$:

$$Q^M(X,X') = \sum_{\lambda}^{N} \mathring{F}'(X,i\kappa_\lambda)|\Gamma_\lambda\rangle \left\langle \Gamma_\lambda \right| \tilde{\mathring{F}}'(X',i\kappa_\lambda)$$
$$- \sum_{\lambda'}^{\mathring{N}} \mathring{F}'(X,i\mathring{\kappa}_{\lambda'})\left|\mathring{\Gamma}_{\lambda'}\right\rangle \langle\mathring{\Gamma}_{\lambda'}| \tilde{\mathring{F}}'(X',i\mathring{\kappa}_{\lambda'}). \quad (66)$$

Accordingly, in the kernel $K^M(X,X')$ (60) there is also an additional sum of factorized terms:

$$K^M(X,X') = -\sum_{\lambda}^{N} |\chi'(X,i\kappa_\lambda\rangle \left\langle \mathring{\chi}'(X',i\kappa_\lambda) \right|$$
$$+ \sum_{\lambda'}^{\mathring{N}} \left|\chi'(X,i\mathring{\kappa}_{\lambda'})\right\rangle \langle\mathring{\chi}'(X',i\mathring{\kappa}_{\lambda'})|. \quad (67)$$

With allowance for these changes in the kernels $Q$ and $K$, the derivation of the relations (64) and (65) is repeated trivially. Cases with a rational Jost matrix can also be reduced to an algebraic procedure. The potential matrices and solutions can be obtained in a similar way in the Gel'fand–Levitan approach and in the $R$-matrix inverse scattering problem.[119]

After the determination of the potential matrix $U'(X)$, using a connecting relation analogous to (17) for the matrix functions

$$\mathring{\chi}'(X,P) = \mathscr{U}(X)\mathring{\chi}(X,P) \quad (68)$$

we find in accordance with the expressions (39), (42), (40), and (43) the potential matrix $U(X)$ of the system of equations (28) with extended derivative and its corresponding matrix of solutions

$$U^s(X)=U(X)-\mathring{U}(X)=2\mathscr{U}^{-1}(X)\times\left(\frac{d}{dX}\mathring{\chi}'^{T}(X)P^{-1}(X)\mathring{\chi}'(X)\right)\mathscr{U}(X) \quad (69)$$

or, in matrix form,

$$U^s_{ij}(X,X')=2\sum_{i'j'}\mathscr{U}^{-1}_{ii'}\left(\frac{d}{dX}\sum_{\nu\lambda}\mathring{\chi}'_i(X,i\kappa_\nu)P^{-1}_{\nu\lambda}(X)\mathring{\chi}'_j\times(X',i\kappa_\lambda)\right)\mathscr{U}_{j'j}(X),$$

$$F_\pm(X,P)=\mathring{F}_\pm(X,P)-\mathring{\chi}^T(X)P^{-1}(X)\times\int_X^\infty\mathring{\chi}(X')\mathring{F}_\pm(X',P)dX', \quad (70)$$

$$P=\mathrm{diag}(P_j),\quad \kappa_\lambda=\mathrm{diag}(\kappa_\lambda)_j,\quad (i\kappa_\lambda)_j=\sqrt{E_\lambda-\mathscr{E}_j}.$$

Note that the transport matrices $\mathscr{U}(X,\mathring{X})$ in (68) and (17) are the same, since they are defined on the same basis functions $|\Phi\rangle$ and $|e\rangle$.

Since the functions of the parametric basis are known and form the complete set (10), we can also determine the multidimensional potential

$$V(X,\hat{\mathbf{X}})=\int\delta(\hat{\mathbf{X}}-\hat{\mathbf{X}}')V(X,\hat{\mathbf{X}},\hat{\mathbf{X}}')d\hat{\mathbf{X}}',$$

by using the expression

$$V^s(X,\hat{\mathbf{X}})=V(X,\hat{\mathbf{X}})-V^f(X,\hat{\mathbf{X}})=2\sum_{ij}^m\Phi_i(X,\hat{\mathbf{X}})\sum_{i'j'}\mathscr{U}^{-1}_{ii'}(X)\frac{d}{dX}\times\left(\sum_{\nu\lambda}\mathring{\chi}'_i(X,i\kappa_\nu)P^{-1}_{\nu\lambda}(X)\mathring{\chi}'_j(X',i\kappa_\lambda)\mathscr{U}_{j'j}(X)\right)\times\Phi^*_j(X,\hat{\mathbf{X}}). \quad (71)$$

In the general case, the potential $V^s$ is nonlocal with respect to the angles $\hat{\mathbf{X}}$ and $\hat{\mathbf{X}}'$:

$$V^s(X,\hat{\mathbf{X}},\mathbf{X}')=\sum_{ij}\Phi_i(X,\hat{\mathbf{X}})U_{ij}(X)\Phi^*_j(X,\hat{\mathbf{X}}').$$

We now consider the more complicated problem of the construction of both $V^s(X,\hat{\mathbf{X}})$ and $V^f(X,\hat{\mathbf{X}})$ from known scattering data or the spectral characteristics of the original multidimensional problem.

### 4.3. Reconstruction of the interaction matrix $V^f$ of the fast dynamics

From the set of scattering data $\{\hat{S}(P),\kappa_\lambda,M_\lambda\}$ we reconstruct the potential matrix $U^{f'}(X)=\mathscr{U}(X)h^f(X)\mathscr{U}^{-1}(X)$ and find solutions $\chi'$ of the system (16) in accordance with the usual or generalized multichannel Gel'fand–Levitan–Marchenko expressions (30)–(32). We then find the bilocal transport operator $\mathscr{U}(X)$ and the terms $\mathscr{E}(X)$ by solving the algebraic eigenvalue problem[11]

$$U^{f'}(X)\mathscr{U}(X)=\mathscr{U}(X)U^f(X)=\mathscr{U}(X)\mathscr{E}(X). \quad (72)$$

Thus, we have obtained a method for finding terms from the solution of the multichannel inverse problem, and not from the direct problem for the reference equation (27), as in the previous case. Knowledge of $\mathscr{U}(X)$ also enables us to reconstruct the matrix of the effective vector potential (7):

$$A(X)=i\mathscr{U}^{-1}(X)\frac{d}{dX}\mathscr{U}(X), \quad (73)$$

which is responsible for the appearance of a velocity-dependent potential.

Using the technique of degenerate kernels presented above, we give an example of an exactly solvable model for the system of equations with respect to the slow variables (16). To achieve the greatest simplicity, we shall in reconstructing $V^f(\mathrm{X})$ take as the reference potential $\mathring{V}(\mathrm{X})=0$. For reflectionless potentials with respect to the slow variable there remains in $Q^M(X,X')$ only a sum over bound states. We restrict ourselves to considering the example of "transparent" potential matrices in Marchenko's approach with one bound state $\lambda=1$, $E_1=-\kappa_j^2+\varepsilon_j$ for the system (16). Then

$$\mathring{F}'_{ij}(i\kappa,X)=\exp(-\kappa_iX)\delta_{ij},\quad \mathring{\chi}'(X,i\kappa_j)=\exp(-\kappa_jX)\gamma_j. \quad (74)$$

Using the relations (61)–(63), we obtain

$$\chi_j(i\kappa,X)=\frac{\exp(-\kappa_jX)\gamma_j}{1+\Sigma_i^m(\gamma_i^2/2\kappa_i)\exp(-2\kappa_iX)}, \quad (75)$$

$$K_{jj'}(X,X')=-\frac{\exp(-\kappa_jX)\gamma_j\gamma_{j'}\exp(-\kappa_{j'}X)}{1+\Sigma_i^m(\gamma_i^2/2\kappa_i)\exp(-2\kappa_iX)}. \quad (76)$$

For the elements of the potential matrix and the solutions, we obtain from the expressions (64) and (65) explicit relations that are the multichannel generalization of the relations for potentials of Eckart type:

$$U^{f'}_{jj'}(X)=2\gamma_j\gamma_{j'}\frac{d}{dX}\left[\frac{\exp(-(\kappa_j+\kappa_{j'})X)}{1+\Sigma_i^m(\gamma_i^2/2\kappa_i)\exp(-2\kappa_iX)}\right], \quad (77)$$

$$F'^{\pm}_{jj'}(k,X)=\exp(\pm ik_jX)\delta_{jj'}-\frac{\gamma_j\gamma_{j'}\exp(-\kappa_jX)\int_X^\infty\exp(-(\kappa_{j'}\pm ik_{j'})X')dX'}{1+\Sigma_i^m(\gamma_i^2/2\kappa_i)\exp(-2\kappa_iX)}. \quad (78)$$

The summation here is over the number $m$ of terms $\mathscr{E}_i(X)$, which can be found by implementing the diagonalization procedure (72) for the potential matrix (77).

We consider, for example, a two-channel exactly solvable model. Let $U'$ be reconstructed in the explicit form (77); the channel indices take only the two values $i$, $j=1$,

2. From the procedure (72) for diagonalizing the potential matrix $U'$, $\mathscr{U}^{-1}(X)U'\mathscr{U}(X)=\mathscr{E}(X)$, we obtain $\mathscr{U}(X)$, $\mathscr{E}(X)$, and $\delta(X)$:

$$\mathscr{U}(X)=\begin{pmatrix} \cos\delta(X) & \sin\delta(X) \\ -\sin\delta(X) & \cos\delta(X) \end{pmatrix},$$

where

$$\delta(X)=\int_X A_{12}(X')dX'.$$

It follows from these relations that

$$\begin{pmatrix} \mathscr{E}_1\cos^2\delta+\mathscr{E}_2\sin^2\delta & (\mathscr{E}_1-\mathscr{E}_2)\cos\delta\sin\delta \\ (\mathscr{E}_1-\mathscr{E}_2)\cos\delta\sin\delta & \mathscr{E}_1\sin^2\delta+\mathscr{E}_2\cos^2\delta \end{pmatrix}$$

$$=\begin{pmatrix} U'_{11} & U'_{12} \\ U'_{21} & U'_{22} \end{pmatrix}.$$

We then have

$$\tan 2\delta(X)=\frac{U'_{21}(X)}{U'_{11}(X)-U'_{22}(X)},$$

$$\mathscr{E}_1(X)=\frac{U'_{11}(X)+U'_{22}(X)}{2}+\frac{U'_{12}(X)}{\sin 2\delta(X)},$$

$$\mathscr{E}_2(X)=\frac{U'_{11}(X)+U'_{22}(X)}{2}-\frac{U'_{12}(X)}{\sin 2\delta(X)}.$$

This is a beautiful model for investigating the problem of term crossing using analytic expressions of the form (77).

To reconstruct the multidimensional potential $V^f(\mathbf{X})$, it is necessary in the second stage, using the spectral data $\{s(X,k),\mathscr{E}_i(X),\gamma_i^2(X)\}$, which are functions of the slow variable $X$, to formulate a parametric inverse problem for the recovery of $V^f(X,.)$ and $\Phi_i(X,.)$ for the fast dynamic equation (4) or (27) for each fixed value of $X$. For the parametric family of inverse problems, one can, as for ordinary problems, develop a technique of Bargmann potentials, making it possible to construct in explicit analytic form the solutions $\Phi_i(X,.)$ and the potential $V^f(X,.)$.

## 5. PARAMETRIC FAMILY OF INVERSE PROBLEMS FOR THE FAST DYNAMIC EQUATION

We consider the parametric formulation of the inverse problem for the example of reconstruction of the two-dimensional potential $V^f(X,y)$. In Cartesian coordinates, we write Eq. (4) with respect to the fast variable $y$ for each fixed value $X\equiv x$ of the slow variable in the form

$$\left[-\frac{d^2}{dy^2}+\mathring{V}^f(y)+V^f(x;y)\right]\psi(x;y)=\mathscr{E}(x)\psi(x;y). \tag{79}$$

Since the completeness of the set of eigenfunctions (10) is important in the formulation of the inverse problem, we use it for the physical normalized eigenfunctions of Eq. (79), which depend parametrically on $x$:

$$\psi(x;k,y)=f_-(x;k,y)-s(x;k)f_+(x;k,y), \tag{80}$$

$$\psi(i\kappa_n(x),y)=\gamma_n(x)f_+(i\kappa_n(x),y), \tag{81}$$

and for the regular solutions

$$\phi(x;k,y)=\frac{1}{2ik}[f_-(x;k)f_+(x;k,y)-f_+(x;k)f_-(x;k,y)]. \tag{82}$$

The $S$ matrix

$$s(x;k)=f_-(x;k)/f_+(x;k) \tag{83}$$

is determined by the Jost functions $f_\pm(x;k)$, which depend on $x$ as on a parameter:

$$f_\pm(x;k)=\lim_{y\to 0} f_\pm(x;k,y). \tag{84}$$

As usual, we determine the normalization $\gamma_n^2(x)$ of the terms $\mathscr{E}_n(x)$ through the Jost solutions for $k=i\kappa_n(x)$. With allowance for (11), we obtain

$$\gamma_n^{-2}(x)=\int_0^\infty |f(i\kappa_n(x),y)|^2dy=\sum_j^m \mathscr{U}_{nj}(x)\int_0^\infty |e_j^M(y)|^2dy\,\mathscr{U}_{jn}(x), \tag{85}$$

or

$$\gamma_n^{-2}(x)=\sum_j^m \mathscr{U}_{nj}(x)M_j^{-2}\mathscr{U}_{jn}(x).$$

Realizing the derivation of the generalized equations of the inverse problem (Refs. 82, 39, 110, 111, and 109) for $\mathring{V}^f(y)$, we obtain the parametric expressions

$$K(x;y,y')+Q(x;y,y')+\int_{y(0)}^{\infty(y)} K(x;y,y')\times Q(x;y'',y')dy''=0, \tag{86}$$

$$V^f(x;y)=\mathring{V}^f(y)\mp 2\frac{d}{dy}K(x;y,y), \tag{87}$$

$$\psi(x;k,y)=\mathring{\psi}(k,y)+\int_{y(0)}^{\infty(y)} K(x;y,y')\mathring{\psi}(k,y')dy'. \tag{88}$$

The limits of integration in (86) and (88) and the signs in (87) depend on the particular approach to the inverse problem. The limits from $y$ to $\infty$ (from $y$ to $a$) in (86) and (88) and the minus sign in (87) correspond to Marchenko's formulation ($R$-matrix inverse problem, Ref. 39). The limits $[0,y]$ in (86) and (88) and plus sign in (87) correspond to the Gel'fand–Levitan approach.

In the framework of the generalized Marchenko approach,[109,110] we construct the integral kernels $Q^M(x;y,y')$, which depend parametrically on $x$,

$$Q^M(x;y,y')=\frac{1}{2\pi}\int_{-\infty}^{\infty}[\mathring{s}(k)-s(x;k)]\mathring{f}(k,y)\mathring{f}(k,y')dk+\sum_n^m \gamma_n^2(x)\mathring{f}(i\kappa_n(x),y)\mathring{f}(i\kappa_n(x),y')$$

$$-\sum_n^{\mathring{m}} \mathring{\gamma}_n^2(x) f(i\mathring{\kappa}_n(x),y)\mathring{f}(i\mathring{\kappa}_n(x),y'), \quad (89)$$

using two sets of scattering data: $\{s(x,k),\mathscr{E}_n(x),\gamma_n^2(x)\}$, which correspond to Eq. (79) for each value of the parameter $x$, and the ordinary scattering data $\{\mathring{s}(,k),\mathring{\mathscr{E}}_n,\mathring{\gamma}_n^2\}$, which correspond to (79) with $V^f(x;y)=0$:

$$\left[-\frac{d^2}{dy^2}+\mathring{V}^f(y)\right]\mathring{\psi}(k,y)=\mathscr{E}(x)\mathring{\psi}(k,y). \quad (90)$$

The functions $\mathring{f}(k,y)$ are the ordinary Jost solutions of Eq. (90) with the known potential $\mathring{V}^f(y)$. From the integral kernels $K^M(x;y,y')$, with respect to which the linear integral equation (86) is solved for each fixed $x$, the potentials (87) and the Jost solutions (88) are determined.

We construct similarly a parametric family of kernels $Q^{\rm GL}(x;y,y')$ of the integral equation that generalizes the Gel'fand–Levitan equation:

$$Q^{\rm GL}(x;y,y')=\int_0^\infty \mathring{\phi}(k,y)\mathring{\phi}(k,y')d[\rho(x;k)-\mathring{\rho}(k)]$$
$$+\sum_n^m c_n^2(x)\mathring{\phi}(i\kappa_n(x),y)\mathring{\phi}(i\kappa_n(x),y')$$
$$-\sum_n^{\mathring{m}} \mathring{c}_n^2(x)\mathring{\phi}(i\mathring{\kappa}_n(x),y)\mathring{\phi}(i\mathring{\kappa}_n(x),y'). \quad (91)$$

The spectral function $\rho(x;k)$ is determined by the Jost functions $f_\pm(x;k)$:

$$\rho(x;k)=\frac{2k}{\pi f_-(x;k)f_+(x;k)}, \quad (92)$$

and the normalization $c_n^2(x)$ by the regular solutions $\phi(x;k,y)$ for $k=i\kappa_n(x)$:

$$c_n^2(x)=\left\{\int_0^\infty |\phi(i\kappa_n(x),y)|^2dy\right\}^{-1}. \quad (93)$$

The connection with the ordinary normalization coefficients is readily established after we have substituted in (93) the relation (11) that connects the moving frame to the fixed frame:

$$c_n^2(x)=\sum_j^m \mathscr{U}_{nj}(x)N_j^{-2}\mathscr{U}_{jn}(x),\ N_j^{-2}=\int_0^\infty |e_j^{\rm GL}(y)|^2dy. \quad (94)$$

The spectral characteristics $\{\mathring{\rho}(k)\mathring{c}_n^2,\mathring{\mathscr{E}}_n\}$ and the regular solutions $\mathring{\phi}(k,y)$ correspond to Eq. (90). After we have solved Eq. (86) for $K^{\rm GL}(x,y,y')$, we use the expressions (87) (lower sign) and (88) (limits [0,x]) to find the potential and regular solutions of Eq. (79).

Similarly, we can derive the relations of the parametric family of inverse problems in $R$-matrix scattering theory.[121,122] It follows from the expressions that we have given that for one-dimensional fast motion we can reconstruct a two-dimensional potential with parametric dependence on the slow variable and as a function of the fast variable. We use here scattering data with parametric dependence on the coordinate variable. In the complete formulation of the inverse problem, when the initial data take the form of a multidimensional amplitude, the dependence of the spectral parameters on the slow variables is determined from (72) after the reconstruction of the potential matrix $U^{f'}(x)$ in the solution of the inverse problem for the system (16), which describes the slow dynamics. One can also consider model problems by specifying a functional dependence of the spectral parameters in order to study the geometrical aspects of multidimensional and many-particle quantum scattering theory. Thus, the use of the technique of the adiabatic representation makes it possible to increase the dimension of the space for which formulation of an inverse problem is possible. In the following section, we shall apply the technique of Bargmann potentials (Refs. 38, 39, 82, and 123) to the parametric family of inverse problems for Eqs. (79).

## 6. BARGMANN POTENTIALS WITH PARAMETRIC DEPENDENCE ON THE SLOW VARIABLE

To the Bargmann potentials for the ordinary one-dimensional Schrödinger equation there correspond rational Jost functions

$$f(k)=\mathring{f}\prod\frac{k-i\alpha}{k+i\beta}. \quad (95)$$

Then the kernels of the integral equations of the inverse problems separate,

$$Q(y,y')=\sum_i^N Q_i(y)Q_i(y'), \quad (96)$$

and the equations themselves reduce to algebraic systems and can be solved explicitly if the reference potentials $\mathring{V}(y)$ admit analytic solutions.

We now construct a large class of potentials for which we can find in a closed analytic form solutions of the parametric Schrödinger equation (79). By analogy with (95), we choose the Jost function in rational form, but it now depends parametrically on the dynamical slow variable $x$ through the dependence on it of the spectral parameters:

$$f(x;k)=\mathring{f}(k)\prod\frac{k-i\alpha(x)}{k+i\beta(x)}. \quad (97)$$

The parametric Jost function (97) has $N$ simple poles at the point $k=i\beta_j(x)$ and $N$ simple zeros at $k=i\alpha_j(x)$. Moreover, $\alpha(x)$ contains not only zeros on the imaginary half-axis, corresponding to bound states, Re $\kappa_j(x)=0$, Im $\kappa_j(x)>0$, but also zeros in the lower half-plane with Im $\nu_j(x)<0$ (the number of simple poles $\beta_j$ is equal to the number of $k_j$ and $\nu_j$ values taken together). Then the $S$ matrix and spectral function take the form

$$s(x;k)=\mathring{s}(k)\prod\frac{(k+i\alpha(x))(k+i\beta(x))}{(k-i\beta(x))(k-i\alpha(x))}, \quad (98)$$

$$\rho(x;k)=\mathring{\rho}(k)\prod\frac{(k-i\beta(x))(k+i\beta(x))}{(k+i\alpha(x))(k-i\alpha(x))}. \quad (99)$$

The functions $\mathring{s}(k)$ and $\mathring{\rho}(k)$ are known, since the potential $\mathring{V}(y)$ is known; in the special case, $\mathring{V}(y)=0$. For $s(x,k)$ and $\rho(x,k)$ such as (98) and (99), the kernels

$Q(x;y,y')$ of the integral equations of the inverse problems degenerate and can be represented as sums of several terms with factorized coordinate dependence with respect to the fast variable:

$$Q(x;y,y')=\sum_{i}^{N}B_i(x;y)B_i(x;y'). \qquad (100)$$

After the substitution of such a kernel $Q$ in the basic parametric equation (86) of the inverse problem, it is obvious that the kernels of generalized shift $K(x;y,y')$ also become degenerate:

$$K(x;y,y')=\sum_{i}^{N}K_i(x;y)B_i(x;y'). \qquad (101)$$

As a consequence of this, the integral equations become algebraic, and the spherically asymmetric potential and the solutions corresponding to it can be expressed in closed form in terms of the known solutions of the nonparametric problem (90) with potential $\mathring{V}(y)$ and in terms of the spectral characteristics of the two problems (90) and (79): the nonparametric one with potential $\mathring{V}(y)$ and the parametric one with potential $\mathring{V}(y)+V(x,y)$. We shall consider several examples of the use of Bargmann potentials.

### 6.1. Determination of analytic solutions in the Gel'fand–Levitan approach

Suppose that for $E>0$ we have $\rho(x;k)=\mathring{\rho}(k)$ and that there is only one bound state $\mathscr{E}(X)=-\kappa^2(x)$. To achieve the maximum simplification of the problem, we choose the reference potential $\mathring{V}^f(y)\equiv 0$. To this case there corresponds the Jost function

$$f(x;k)=\frac{k-i\kappa(x)}{k+i\kappa(x)}. \qquad (102)$$

Hence, as follows from (74), $\rho(x,k)=2k/\pi=\mathring{\rho}(k)$. Then from the relation (87) we obtain

$$Q^{\rm GL}(x;y,y')=c^2(x)\frac{\sinh[\kappa(x)y]\sinh[\kappa(x)y']}{\kappa^2(x)}, \qquad (103)$$

$$K^{\rm GL}(x;y,y')=-c^2(x)\phi(i\kappa(x),y)\frac{\sinh[\kappa(x)y']}{\kappa(x)}, \qquad (104)$$

$$\phi(i\kappa(x),y)=\frac{\kappa(x)\sinh(\kappa(x)y)}{\kappa^2(x)+1/2c^2(\sinh(2\kappa(x)y)/2\kappa(x))-y}. \qquad (105)$$

Finally, taking into account (105) in (104) and substituting the result in the relations (88) and (87) for $\mathring{V}(y)=0$, we obtain in explicit form expressions for the solutions for arbitrary $k$ and for a two-dimensional potential $V(x;y)$ with parametric dependence on $x$:

$$\phi(x;k,y)=\frac{\sin ky}{k}-c^2(x)\phi(i\kappa(x),y)\int_0^y\frac{\sinh[\kappa(x)y']\sin ky'dy'}{\kappa(x)k}$$
$$=\frac{\sin ky}{k}-\frac{c^2(x)\sinh(\kappa(x)y)[\kappa(x)\cosh(\kappa(x)y)\sin ky-k\sinh(\kappa(x)y)\cos ky]}{k[\kappa^2(x)+k^2][\kappa^2(x)+1/2c^2(x)(\sinh(2\kappa(x)y)/2\kappa(x))-y]}, \qquad (106)$$

$$V(x;y)=\frac{2\kappa(x)[y/2-c^{-2}(x)\kappa^2(x)]-2\sinh^2(\kappa(x)y)}{[c^{-2}(x)\kappa^2(x)+1/2[\sinh(2\kappa(x)y)/2\kappa(x)]-y]^2}. \qquad (107)$$

We can verify the method of construction as follows. Going over in (106) to the asymptotic form as $y\to\infty$ and separating the terms with $\exp(\pm iky)$, we find the coefficients of them. These are the Jost functions $f_\pm(x;k)$, which are identical to (102). From them, using the definition (83), we construct the parametric $s$ matrix

$$s(x;k)=\frac{f_-(x;k)}{f_+(x;k)}=\frac{(k+i\kappa(x))^2}{(k-i\kappa(x))^2}. \qquad (108)$$

To the example with $m$ bound states and potential $\mathring{V}(y)\neq 0$ there corresponds

$$f_+(x;k)=\mathring{f}(k)\prod_{n}^{m}\frac{k-i\kappa_n(x)}{k+i\kappa_n(x)}, \qquad (109)$$

where the terms $\mathscr{E}_n(x)=-\kappa_n^2(x)$ are found by solving the multichannel system of equations with respect to the slow variables and the subsequent diagonalization (72). It is easy to write down the $m$-term generalization of the expressions (103)–(107). The kernels of the basic Gel'fand–Levitan equation (86) are written in the form

$$Q^{\rm GL}(x;y,y')=\sum_{n}^{m}c_n^2(x)\mathring{\phi}[i\kappa_n(x),y]\mathring{\phi}[i\kappa_n(x),y'], \qquad (110)$$

$$K^{\rm GL}(x;y,y')=-\sum_{n}^{m}c_n^2(x)\phi(i\kappa_n(x),y)\mathring{\phi}[i\kappa_n(x),y']. \qquad (111)$$

After this, using one of the relations (86) or (88) of the inverse problem, we obtain the solutions $\phi[i\kappa_n(x),y]$ for the bound states $\mathscr{E}_n(x)=-\kappa_n^2(x)$; they depend parametrically on the dynamical variable $x$:

$$\phi[i\kappa_n(x),y]=\sum_j^m \mathring{\phi}[i\kappa_j(x),y]P_{jn}^{-1}(x;y), \qquad (112)$$

where

$$P_{nj}(x;y)=\delta_{nj}+c_n^2\int_0^y \mathring{\phi}[i\kappa_n(x),y']\mathring{\phi}[i\kappa_j(x),y']dy'.$$

Then, substituting the expression obtained for $\phi[i\kappa_n(x),y]$ in the relation (112) for the kernel $K^{\rm GL}(x;y,y')$ and using the relations (86)–(88) of the parametric inverse problem, we obtain in closed analytic form expressions for the potential and solutions:

$$V(x;y)=-2\frac{d^2}{dy^2}\ln\det\|P_{nj}(x;y)\|,$$

$$\phi(x;k,y)=\mathring{\phi}(k,y)-\sum_n^m\sum_j^m \mathring{\phi}[i\kappa_j(x),y]P_{jn}^{-1}(x;y)\times\int_0^y \mathring{\phi}[i\kappa_n(x),y']\mathring{\phi}(k,y')dy'. \qquad (113)$$

The investigations are readily made in spherical coordinates by choosing as the fast variable the angle and as the slow variable the coordinate and vice versa.

### 6.2. Determination of analytic solutions in Marchenko's approach

To reflectionless (transparent) potentials with respect to the fast variable there corresponds a one-dimensional inverse problem on the complete axis $-\infty<y<\infty$ with reflection coefficient equal to zero: $s^{\rm ref}=0$. The transmission coefficient $s^{\rm tr}$, which has absolute value 1, has the form of a rational fraction:

$$s^{\rm tr}(x;k)=\prod_n^m\frac{k+i\kappa_n(x)}{k-i\kappa_n(x)}. \qquad (114)$$

We note that different forms are possible: potentials that are transparent with respect to both the slow and the fast variable, transparent along one of them, trapping along one of the coordinates or both, and there are similar treatments with rational Jost functions.

The inverse problem on the complete axis is similar to a two-channel problem with two coupled basic integral equations. However, since $V(x,y)$ can be expressed in terms of the kernel $K_1(x;y,y')$ of the one integral equation as well as in terms of the kernel $K_2(x;y,y')$ of the other, it is sufficient to find one of the $K_i(x;y,y')$ using an expression identical to (86). Then in $Q^M(x;y,y')$, which is determined here for $\mathring{V}(y)\equiv 0$ in accordance with

$$Q^M(x;y,y')=\frac{1}{2\pi}\int_{-\infty}^{\infty}s(x;k)\exp(ik(y+y'))dk+\sum_n^m\gamma_n^2(x)\exp(-\kappa_n(x)(y+y')), \qquad (115)$$

there remains only the contribution from the states of the discrete spectrum:

$$Q^M(x;y,y')=\sum_n^m\gamma_n^2(x)\exp(-\kappa_n(x)(y+y')). \qquad (116)$$

Similarly, for $K^M(x;y,y')$ we have

$$K^M(x;y,y')=-\sum_n^m\gamma_n^2(x)f(i\kappa_n(x),y)\exp(-\kappa_n(x)y'). \qquad (117)$$

For the Jost solutions for $k=i\kappa_n(x)$ we obtain from (86) the system of algebraic equations

$$f(i\kappa_n(x),y)=\sum_j^m\exp(-\kappa_j(x)y)P_{jn}^{-1}(x;y) \qquad (118)$$

with a matrix of coefficients $P_{jn}(x;y)$ that depend parametrically (through the spectral parameters) on $x$:

$$P_{jn}(x;y)=\delta_{nj}+\frac{\gamma_n^2(x)\exp[-(\kappa_n(x)+\kappa_j(x))y]}{\kappa_n(x)+\kappa_j(x)}. \qquad (119)$$

Substituting $f(i\kappa_n(x),y)$ in $K^M(x;y,y')$ (117) and using (87) and (88), we obtain

$$V(x;y)=-2\frac{d^2}{dy^2}\ln\det\|P_{nj}(x;y)\|, \qquad (120)$$

$$f_\pm(x;k,y)=\exp(\pm iky)+\sum_{nj}\gamma_n^2(x)\exp(-\kappa_n(x)y)\times P_{nj}^{-1}(x;y)\frac{\exp((-\kappa_j(x)\pm ik)y)}{\kappa_j(x)\mp ik}. \qquad (121)$$

In the case of one bound state, we obtain an expression for the generalized Eckart potential:

$$V(x;y)=-2\frac{2\kappa(x)\gamma^2(x)\exp(-2\kappa(x)y)}{1+(\gamma^2(x)/2\kappa(x))\exp(-2\kappa(x)y)}, \qquad (122)$$

which can be transformed to a simpler form by using the substitution $\exp(2\kappa(x)y_0)=\gamma^2(x)/2\kappa(x)$ and the transformation

$$\{1+\exp[-2\kappa(x)(y-y_0(x))]\}^2=4\cosh^2[\kappa(x)(y-y_0(x))]\exp[-2\kappa(x)(y-y_0(x))], \qquad (123)$$

$$V(x;y)=-\frac{\kappa^2(x)}{\cosh^2[\kappa(x)(y-y_0(x))]}.$$

The Jost solutions that correspond to it on the trajectory of the varying bound state $-\kappa^2(x)$, and also for arbitrary values of $k$, can be written in the explicit form

$$f(i\kappa(x),y)=\frac{\exp(-\kappa(x)y)}{\exp[-2\kappa(x)(y-y_0(x))]}, \qquad (124)$$

$$f_{\pm}(x;k,y)$$
$$=\exp(\pm iky)$$
$$\times\left\{1-\frac{\exp[-2\kappa(x)(y-y_0(x))]}{1+\exp[-2\kappa(x)(y-y_0(x))](\kappa(x)\mp ik)}\right\}. \quad (125)$$

### 6.3. Parametric family of phase-equivalent potentials

The procedure for constructing phase-equivalent potentials can be realized for a parametric inverse problem in which the spectral characteristics depend on an external coordinate variables as on a parameter: $\{M_n^2(x),\kappa_n(x), b_n(x),S(x,k);C_n^2(x),\rho(x;k)\}$. The dependence on the slow coordinate $x$ is determined by the operator $\mathscr{U}(x,\mathring{x})$ of parallel transport of the frame. In the $S$ matrix corresponding to the gauge equation (6) there may arise poles manifested in the form of geometric phases associated with the singularities in the behavior of the connection $A$ induced by the functions of the basis parametric equation.

For a parametric inverse problem that is rational or on the half-axis with Jost function (97) generalizing (95),

$$f(x;k)=\prod_n^N\frac{k-i\kappa_n(x)}{k+ib_n(x)}, \quad (126)$$

the $S$ matrix can be written in the form

$$S(x;k)=\prod_n^N\frac{k+i\kappa_n(x)}{k-ib_n(x)}\frac{k+ib_n(x)}{k-i\kappa_n(x)}. \quad (127)$$

The kernel of the basic integral equation (86) in Marchenko's approach,

$$Q(x;y,y')=\frac{1}{2\pi}\int_{-\infty}^{\infty}[1-S(x;k)]\exp[ik(y+y')]$$
$$+\sum_n^N M_n^2\exp[-\kappa_n(y+y')], \quad (128)$$

can be rewritten with allowance for (127) in the form

$$Q(x;t=y+y')=-i\sum_n^N\operatorname{Res}S(k=ib_n(x))\exp[-b_n(x)t]$$
$$-i\sum_n^N\operatorname{Res}S(k=i\kappa_n(x))\exp[-\kappa_n(x)t]$$
$$+\sum_n^N M_n^2\exp(-\kappa_n(x)t). \quad (129)$$

Setting in the first stage

$$\mathring{M}_n^2=i\operatorname{Res}S(k)|_{k=i\kappa_n(x)}, \quad (130)$$

we obtain

$$\mathring{Q}(x;t)=-i\sum_n^N\operatorname{Res}S(k)|_{k=ib_n(x)}\exp[-b_n(x)t]$$
$$\equiv A_n(x)\exp[-b_n(x)t], \quad (131)$$

where

$$A_n(x)=\frac{2b_n(x)(b_n(x)+\kappa_n(x))}{(b_n(x)-\kappa_n(x))}$$
$$\times\prod_{n'\neq n}^N\frac{(b_n(x)+\kappa_n'(x))(b_n(x)+b_n'(x))}{(b_n(x)-b_n'(x))(b_n(x)-\kappa_n'(x))}. \quad (132)$$

The corresponding expressions of the Bargmann approach are obtained if we set $\kappa_n(x)\equiv\kappa_n$ and $b_n(x)\equiv b_n$.[39,123] Substituting the kernel $\mathring{Q}(x;y,y')$ (131) in the parametric equations of the inverse problem (86)–(88), we obtain

$$\mathring{V}(x;y)=-2\frac{d^2}{dy^2}\ln\det\|P(x;y)\|; \quad (133)$$

$$\mathring{f}^{\pm}(x;k,y)=\exp(\pm iky)+\sum_{nn'}^N A_n(x)P_{nn'}^{-1}(x;y)$$
$$\times\frac{\exp[-(b_n(x)+b_{n'}(x)\mp ik)y]}{(b_n(x)\mp ik)}. \quad (134)$$

Here, $P_{nn'}(x;y)$ is determined as follows:

$$P_{nn'}(x;y)=\delta_{nn'}+A_n(x)\frac{\exp[-(b_n(x)+b_{n'}(x))y]}{(b_n(x)+b_{n'}(x))}. \quad (135)$$

In the second stage, we find the family of potentials and solutions for the normalization constants $i\operatorname{Res}S(k=i\kappa_n(x))<M_n^2<\infty$, which do not satisfy the condition (130). We now obtain an analog of a phase-equivalent family of potentials for the parametric inverse problem. In this case, the kernel of the integral equation (86) can be written in the form

$$Q(x;y,y')=\sum_n^N(M_n^2(x)-\mathring{M}_n^2(x))$$
$$\times\mathring{f}(i\kappa_n(x),y)\mathring{f}(i\kappa_n(x),y'). \quad (136)$$

Similarly, for the kernel of the generalized shift $K^M(x;y,y')$ we have

$$K^M(x;y,y')=-\sum_n^N(M_n^2(x)-\mathring{M}_n^2(x))$$
$$\times f(i\kappa_n(x),y)\mathring{f}(i\kappa_n(x),y'). \quad (137)$$

Substituting $K^M(x;y,y')$ and $F(x;y,y')$ in the basic parametric Marchenko equations (86)–(88), we obtain for the potentials and the Jost solutions the relations

$$V(x;y)=\mathring{V}(x;y)+2\frac{d^2}{dy^2}\ln\det P(x;y), \quad (138)$$

$$f^{\pm}(x;k,y)=\mathring{f}^{\pm}(x;k,y)-\sum_{nm}^N(M_n^2-\mathring{M}_n^2)\mathring{f}(i\kappa_n(x),y)$$
$$\times P_{nm}^{-1}(x;y)\int_r^{\infty}\mathring{f}(i\kappa_m(x),y')\mathring{f}^{\pm}(k,y')dy', \quad (139)$$

in which *the explicit dependence on the fast variables is given by the Jost solutions (134) determined on the potential* *curves—the terms $\kappa_n^2(x)$, which depend parametrically on the slow dynamical variable x.* Here, we have used the notation

$$P_{nm}(x;y)=\delta_{nm}+(M_n^2(x)-\mathring{M}_n^2(x))\int_r^\infty \mathring{f}(i\kappa_n(x)y')\times\mathring{f}(i\kappa_m(x),y')dy'. \tag{140}$$

In this section, we have demonstrated some examples of exactly solvable parametric models in order to show how the technique of Bargmann potentials can be extended to a parametric family of inverse problems. *Specifying the functional dependence of the spectral characteristics on the external coordinate variable, we obtain a large class of exactly solvable multidimensional models on the basis of a parametric inverse problem for equations of lower dimension.* Such a formulation can be used in the search for analytic solutions of nonlinear evolution equations. Thus, the parametric inverse problem has independent interest and not only as a constituent part in the solution of the original multidimensional problem in the adiabatic approach.

## 7. THREE-PARTICLE INVERSE SCATTERING PROBLEM

In accordance with the general definition, a three-particle inverse scattering problem consists of the reconstruction of the effective interaction potential from a known scattering amplitude. As is well known, the differential formulation of the modified Faddeev equations is the main tool in a correct numerical investigation of scattering processes in a system of three nonrelativistic particles.[37] In the configuration space of the relative motion of the three particles $\mathbf{X}=\mathbf{x}_\alpha+\mathbf{y}_\alpha\in\mathbf{M}$, where $\alpha=1, 2, 3$ is the index of the pair of particles associated with the coordinate $\mathbf{x}_\alpha$, the equations for the components $F_\alpha(\mathbf{X})$ have the form

$$\left\{-\Delta_{\mathbf{X}}+\hat{V}_\alpha(x_\alpha)+\sum_\beta V_\beta^{(0)}-E\right\}F_\alpha=-\hat{V}_\alpha\sum_{\beta\neq\alpha}F_\beta, \tag{141}$$

where the functions $\hat{V}_\alpha$ and $\mathring{V}_\alpha$ are the short- and long-range parts of the central two-body potentials $V_\alpha=\hat{V}_\alpha+\mathring{V}_\alpha$, $E=k_\alpha^2+p_\alpha^2$ is the c.m.s. energy, and $\mathbf{P}=\{\mathbf{k}_\alpha,\mathbf{p}_\alpha\}$ corresponds to $\mathbf{X}=\{\mathbf{x}_\alpha,\mathbf{y}_\alpha\}$. The asymptotic boundary conditions corresponding to all possible scattering processes in the three-particle system follow from the compact integral equations (141). The corresponding solution of the original Schrödinger problem

$$H\Psi=E\Psi,\quad H=-\Delta_{\mathbf{X}}+V_\alpha(x_\alpha) \tag{142}$$

is specified by the sum of Faddeev components

$$\Psi(\mathbf{X})=\sum_\alpha F_\alpha(\mathbf{X}_\alpha(\mathbf{X})). \tag{143}$$

i) We choose some representation in $\mathbf{M}:\mathbf{X}\in\mathbf{R}_+^1\otimes\hat{\mathbf{M}}$, in which the values $X$ of the base $B$ are given in terms of the first linear invariant of the inertia tensor $X^2=x_\alpha^2+y_\alpha^2$, which determines the hyperradius $X=\sqrt{X^2}$ of the "sphere" $\hat{\mathbf{M}}:\mathbf{S}_X^5(\hat{\mathbf{X}}_\alpha)$.

ii) Using the invariant adiabatic variable $X\in B$, which does not depend on the choice of the Jacobi pair $\alpha$ of coordinates, we introduce for the Faddeev components $F_\alpha(\mathbf{X})$ the following local adiabatic expansions:[7,10]

$$F_\alpha(\mathbf{X})=\sum_j F_{\alpha j}(X;\hat{\mathbf{X}}_\alpha)X^{-1}\chi_j(X). \tag{144}$$

The expansion coefficients $\chi$ are universal for all Faddeev components. The basis components $F_{\alpha j}(X;\hat{\mathbf{X}}_\alpha)$ are determined as solutions of the spectral problem for the system of equations (141) on the sphere $\hat{\mathbf{M}}$ for fixed $X\in B$:

$$\left\{-X^{-2}\Delta_{\hat{\mathbf{X}}_\alpha}+\hat{V}_\alpha(x_\alpha)+\sum_\beta V_\beta^{(0)}-\mathscr{E}_j(X)\right\}F_{\alpha j}(X,\hat{\mathbf{X}}_\alpha)=-\hat{V}_\alpha\sum_{\beta\neq\alpha}F_{\beta j}(X,\hat{\mathbf{X}}_\beta). \tag{145}$$

Here, $\Delta_{\hat{\mathbf{X}}_\alpha}$ is the angle part of the Laplace–Beltrami operator $\Delta_{\mathbf{x}}$ on $\hat{\mathbf{M}}$, and $\mathscr{E}_j(X)$ are eigenvalues; $j$ is the set of quantum numbers that label the spectrum $\sigma(H^f)$ of the three-particle Hamiltonian

$$H^f(X)=-X^{-2}\Delta_{\hat{\mathbf{X}}_\alpha}+\sum_\alpha V_\alpha(X,\hat{\mathbf{X}}_\alpha), \tag{146}$$

which acts on the fiber $\mathscr{F}_X$ and depends on the point $X$ of the base $B$ as on a parameter. The solutions of the Schrödinger problem (142) on $\hat{\mathbf{M}}$,

$$\{H^f(X)-E_j(X)\}\Phi_j(X,\hat{\mathbf{X}})=0, \tag{147}$$

with suitable asymptotic conditions with respect to the parameter $X$ are, apart from the normalization, identical to the functions formed like (143) from a sum of basis components $\{F_{\alpha j}\}$,

$$\Phi_j(X,\hat{\mathbf{X}})=\sum_\alpha^3 F_{\alpha j}(X,\hat{\mathbf{X}}_\alpha(\hat{\mathbf{X}})), \tag{148}$$

and thus inherit the correct asymptotic behavior of the basis Faddeev components $\{F_{\alpha j}\}$ as $X\to\infty$. At the same time, they form in $\mathscr{F}_X=L^2(\hat{\mathbf{M}},d\hat{\mathbf{M}}_X)$ a complete system of orthonormal functions:

$$\langle i|j\rangle=\int\{\Phi_i^*(X;\hat{\mathbf{X}})\Phi_j(X;\hat{\mathbf{X}})d\hat{\mathbf{M}}_X\}=\delta_{ij}, \tag{149}$$

$$X^{2\gamma}\sum_j\Phi_j(X;\hat{\mathbf{X}})\Phi_j^*(X;\hat{\mathbf{X}}')=\delta(\hat{\mathbf{X}},\hat{\mathbf{X}}')/\Omega. \tag{150}$$

The scalar product $\langle\cdot|\cdot\rangle$ in $\mathring{\mathscr{F}}_X$ is defined with $O(5)$-invariant measure $d\hat{M}_X=X^{2\gamma}d\Omega(\hat{\mathbf{X}})$ (with constant $\gamma=3/2$). On the basis of the expansions (143) and (144), and also taking into account (148), we obtain an adiabatic expansion of the three-particle wave function $\Psi(\mathbf{X})$ of the original Schrödinger equation (142) in the form

$$\Psi_i(\mathbf{X},P)=\sum_{\beta j}F_{\beta j}(X;\hat{\mathbf{X}}_\beta(\hat{\mathbf{X}}))X^{-1}\chi_{ji}(X,P)=\sum_j\Phi_j(X;\hat{\mathbf{X}})X^{-1}\chi_{ji}(X,P), \tag{151}$$

$$\Psi(\mathbf{X},\mathbf{P})=(2/\pi)^{1/2}\sum_i \Psi_i(\mathbf{X},P)\Phi_i^*(P;-\hat{\mathbf{P}})P^{-1}. \quad (152)$$

Such a procedure corresponds to a dimensional reduction of the space realized by partial expansions of the original wave function and reduction to problems of lower dimension.

The use of the above definitions makes it possible to introduce the Hilbert bundle $\mathscr{H}(B,\mathscr{F}_{X,\pi})$ with base $B=\mathbf{R}^1_+\ni X$ and unfixed standard fibers $\mathscr{F}_X=L_2(\hat{\mathbf{M}},d\hat{\mathbf{M}})$ formed from real-analytic eigenfunctions $\Phi_j(X;\hat{\mathbf{X}})$ of the operator $H^f(X)$. The fibers $\mathscr{F}_X$ form a family of Hilbert spaces parametrized by points $X\in B$. In the traditional approach of the method of strong channel coupling[51] such a construction corresponds in the expansion of $\Psi(\mathbf{X})=\Sigma_j\Phi_j(Y)\chi_j(X)$ with respect to basis states to a fixed fiber of known basis functions $\Phi_j(Y)$, and the unknown expansion coefficients $\chi_j(X)$, with respect to which the problem is solved, are specified on a base $B$ of dimension lower than the original space. Substitution of the expansion (151) in (142) leads to a system of ordinary differential equations for the coefficients $\chi_j=\{\chi^L_{ji}(X,P)\}$ that is identical to (6),

$$\left[-\sum_j D^2_{ij}(X)+U^L_{ij}(X)-P^2\delta_{ij}\right]\chi^L_{ji}(X)=0, \quad P=\mathrm{diag}(p_n), \quad (153)$$

and makes it possible to find a unique solution of the original problem (141)–(143). The index $L$ denotes a fixed value of the total orbital angular momentum of the corresponding physical quantities; for convenience, it is often omitted. The summation in (153) is with respect to the complete set of channel states. The potential matrix

$$U^L_{ij}(X)=(\mathscr{E}^L_i(X)+\gamma(\gamma+1)X^{-2})\delta_{ij} \quad (154)$$

is formed by the elements of the diagonal matrix of effective potentials determined by the eigenvalues $\mathscr{E}^L_i(X)$ of the fast Hamiltonian $H^f$ (146) and by the centrifugal potential

$$\gamma(\gamma+1)X^{-2}=15/4X^{-2},$$

which is nonvanishing at a triple-collision point. As we saw in Sec. 2, it is determined by the dimension of the space. The covariant derivative $D(X)$ is

$$D_{ij}(X)=\delta_{ij}\partial_X-iA_{ij}(X),$$

where $A_{ij}(X)$ are the matrix components of the connection operator in the Hilbert bundle $\mathscr{H}$, and in addition ensure consistency of the solutions of (153) with the asymptotic boundary conditions on the Faddeev components $F_\alpha(\mathbf{X})$:

$$A_{ij}(X)=\langle\Phi_i|i\partial_X|\Phi_j\rangle+\gamma X^{-1}\delta_{ij}. \quad (155)$$

Thus, the expansion (151) is, by virtue of the use in (142)–(151) of the asymptotic boundary conditions that follow from the compact integral equations, a generalization of the well-known hyperspherical adiabatic expansion.[2,44,54] In addition, the representation (148) was used in Refs. 7, 8, and 12 to determine a new global adiabatic basis $\{\Phi_i\}$, which is constructed in such an approach by means of local basis components $\{F_{\alpha i}\}$, which are solutions of the eigenvalue problem for the Faddeev equations (145) for some fixed $X\in B$. As we see, in the adiabatic representation the coupling between the channels is realized by the matrix elements of the effective gauge field $A(X)$, in contrast to the interaction representation, in which the coupling is realized by the matrix elements of the potential energy. As a result, the adiabatic approach gives a gauge-invariant treatment of multichannel scattering theory in the three-particle system.

### 7.1. Transport operator and system of radial equations

We consider in more detail the scattering problem in a system of three spinless nuclear particles with two-body real-analytic central potentials $V_\alpha\equiv\hat{V}=V^n_\alpha$ that satisfy the conditions[38]

$$\int_0^\infty |V_\alpha(x_\alpha)|dx_\alpha<\infty, \quad \int_0^\infty x_\alpha|V_\alpha(x_\alpha)|dx_\alpha<\infty,$$

and preserve the representation of the total orbital angular momentum:

$$L=l_\alpha+\lambda_\alpha,$$

where $l_\alpha=-ix_\alpha\wedge\nabla_{x\alpha}$ and $\lambda_\alpha=-iy_\alpha\wedge\nabla_{y\alpha}$ are the orbital angular momentum of $\alpha$ and of the third particle. In what follows, we shall, as a rule, omit the set of exact quantum numbers $L=\{L,M,\xi\}$, which are the total angular momentum, its projection, and the total parity $\xi=(-1)^{l_\alpha+\lambda_\alpha}$. The modified equations (141) become the standard Faddeev equations

$$\{-\Delta_{\mathbf{X}_\alpha}+V_\alpha(x_\alpha)-E\}F_\alpha(\mathbf{X}_\alpha,\mathbf{P}) = -\hat{V}_\alpha(x_\alpha)\sum_{\beta\neq\alpha}F_\beta(\mathbf{X}_\beta,\mathbf{P}). \quad (156)$$

Accordingly, the term $V^{(0)}_\beta$ also vanishes on the left-hand side of Eq. (145), which can be rewritten in the form

$$\{-X^{-2}\Delta_{\hat{\mathbf{X}}_\alpha}+V_\alpha(X,\hat{\mathbf{X}}_\alpha)-\mathscr{E}_j(X)\}F_{\alpha j}(X,\hat{\mathbf{X}}_\alpha) = -V_\alpha\sum_{\beta\neq\alpha}F_{\beta j}(X,\hat{\mathbf{X}}_\beta). \quad (157)$$

For definiteness, we choose the two-body potentials $V_\alpha\equiv V^n_\alpha=V_\alpha(X,\hat{\mathbf{X}}_\alpha)$ and homogeneous boundary conditions for Eqs. (145) and (147). Then the complete spectrum $\mathscr{E}(X)=\mathrm{diag}\{\mathscr{E}_j(X)\}\in\sigma(H^f(X))$ of the parametric Hamiltonian $H^f(X)$ (146) is purely discrete and real-analytic for all $X\in(0,\infty)$. Note that the eigenvalues $\mathscr{E}(X)$ of the fast Hamiltonian $H^f(X)$,

$$\mathscr{E}(X)=\langle\Phi(X,\hat{\mathbf{X}})|H^f(X)|\Phi(X,\hat{\mathbf{X}})\rangle, \quad (158)$$

are effective potentials $U(X)$ in the radial equations (153). Therefore, the last requirement is natural from the point of view of scattering theory for potentials that have an analytic continuation to the complex plane of $X$ values. In such an approach, the states both above and below the three-particle breakup threshold are described by a single form, and it is this that determines the advantage of the approach proposed and developed in Refs. 6–10 and 12 compared with the one of Refs. 2, 41, 44, and 54. For greater clarity, we divide the spectrum $\sigma(H^f(X))$ into two parts:

$$\mathscr{E}(X)=\begin{pmatrix}\mathscr{E}_+(X) & 0\\ 0 & \mathscr{E}_-(X)\end{pmatrix} \tag{159}$$

in accordance with the different asymptotic behaviors of the terms:

$$\mathscr{E}_+(X)=\text{diag}\{\mathscr{E}_{i+}(X)\}\in\sigma_+(H^f(X)),$$
$$\text{if } \mathscr{E}_i(X\to\infty)\geqslant 0;$$
$$\mathscr{E}_-(X)=\text{diag}\{\mathscr{E}_{i-}(X)\}\in\sigma_-(H^f(X)),$$
$$\text{if } \mathscr{E}_i(X\to\infty)<0.$$

Therefore, for the complete set of real-analytic functions $\{\Phi\}$ we can introduce the decomposition $\Phi=\Phi_++\Phi_-$, separating explicitly the states $\Phi_+$ corresponding to surface breakup functions and the states $\Phi_-$ corresponding to cluster surface functions. Then the Hilbert fibers $\mathscr{F}_X$ can also be represented as a direct sum $\mathscr{F}=\mathscr{F}_+\dotplus\mathscr{F}_-$. Using the orthogonality and completeness relations (149) and (150), we determine the fixed frame

$$|e(\hat{\mathbf{X}})\rangle\equiv|\Phi(\mathring{X},\hat{\mathbf{X}})\rangle,$$

choosing some fixed point $X=\mathring{X}\in B$ of the base of the bundle $\mathscr{H}(B,\mathscr{F},\pi)$, where the projection $\pi$ is trivial. For each pair of points in $B$ we can now introduce the unitary bilocal operator $\mathscr{U}(X,\mathring{X})$, which maps from $\mathscr{F}_{\mathring{X}}$ to $\mathscr{F}_X$ and realizes parallel transport of the frame from $\mathring{X}$ to the arbitrary point $X$:

$$|\Phi(X,.)\rangle=|e\rangle\mathscr{U}(X,\mathring{X}),\quad \mathscr{U}(X,\mathring{X})=\langle e|\Phi(X,.)\rangle, \tag{160}$$

$$\mathscr{U}(X,\mathring{X})=P\exp i\int_X^{\mathring{X}}A(X')dX'. \tag{161}$$

The structure of the fibers $\mathscr{F}$ in direct-sum form induced by the spectral projectors $Q_\pm=\Sigma_{j\pm}|\Phi_{j\pm}\rangle\langle\Phi_{j\pm}|$ can be naturally generalized to the complete Hilbert bundle $\mathscr{H}$:

$$\pi\Pi_\pm=Q_\pm,$$

where $\Pi_\pm=(\sigma_1\pm i\sigma_2)/2$, and $\sigma_1$ and $i\sigma_2$ are Pauli matrices. For the matrix elements of the transport operators $\mathscr{U}$, we obtain accordingly

$$\mathscr{U}=\begin{pmatrix}\mathscr{U}_{++} & \mathscr{U}_{+-}\\ \mathscr{U}_{-+} & \mathscr{U}_{--}\end{pmatrix}. \tag{162}$$

We rewrite the expansion (151) for the partial wave function $\Psi_i$, also separating the breakup states $\Psi_i^+$ from the cluster states $\Psi_i^-$:

$$\Psi_i(X,\hat{\mathbf{X}})=\Psi_i^+(X,\hat{\mathbf{X}})+\Psi_i^-(X,\hat{\mathbf{X}})$$
$$=\sum_{j+}\Phi_{j+}(X,\hat{\mathbf{X}})X^{-1}\chi_{j+i}(X,P)$$
$$+\sum_{j-}\Phi_{j-}(X,\hat{\mathbf{X}})X^{-1}\chi_{j-1}(X,P). \tag{163}$$

The radial function $\chi\equiv\{\chi_{ji}(X,P)\}$ can be represented in the block form

$$\chi=\begin{pmatrix}\chi_{++} & \chi_{+-}\\ \chi_{-+} & \chi_{--}\end{pmatrix},$$

where each block is a matrix

$$\chi_{++}=[\chi_{j+i+}];\quad \chi_{+-}=[\chi_{j+i-}];$$
$$\chi_{-+}=[\chi_{j-i+}];\quad \chi_{--}=[\chi_{j-i-}].$$

The diagonal matrix of angular momenta is also specified:

$$P=\begin{pmatrix}P_+ & 0\\ 0 & P_-\end{pmatrix}=\begin{pmatrix}\sqrt{E\cdot I_+-\mathscr{E}_+(\infty)} & 0\\ 0 & \sqrt{E\cdot I_--\mathscr{E}_-(\infty)}\end{pmatrix}. \tag{164}$$

Here, $P_+=\text{diag}\,P_{j+}$ is the diagonal matrix of angular momenta of the three free particles. The diagonal matrix $P_-=\text{diag}\,P_{j-}$ corresponds to cluster states of the two-particle channels of the three-particle system,

$$P_-=\sqrt{E\cdot I_--\mathscr{E}_-(\infty)}=\sqrt{p_A^2+\varepsilon_A-\mathscr{E}_-(\infty)}$$
$$=\sqrt{p_A^2+\kappa_A^2-\mathscr{E}_-(\infty)},$$

where $\varepsilon_A=-\kappa_A^2<0$ are the energy eigenvalues of the two-body Hamiltonian

$$h_\alpha=-\Delta_{\mathbf{x}_\alpha}+V_\alpha(x_\alpha).$$

The value of the total energy $E=P_-^2+\mathscr{E}_-(\infty)=p_A^2+\varepsilon_A$ for zero angular momentum $p_A$ of the relative motion of pair $\alpha$ and the scattered particle determines the threshold of the two-particle channel:

$$E_A(0)=-\kappa_A^2<0.$$

The two-particle channel is open when the energy of the system is above the threshold value:

$$E_A(p_A)\geqslant E_A(0),\quad \text{i.e., } p_A^2\geqslant E+\kappa_A^2.$$

The principal channel of decay into three particles opens at zero energy:

$$E_0(0)=0,\quad \text{i.e., } p_A^2=E_+\geqslant 0.$$

Substitution of (152) in the Schrödinger equation (142) and projection onto the complete system of functions $\Phi$ in accordance with (160) makes it possible to realize a $2\times2$ block decomposition in the system of equations (153). Taking into account the block structure of the operator (162), we write down the explicit block representation (153):

$$\langle\Phi|\{X^{-5}\partial_X X^5\partial_X+H^f(X;\hat{\mathbf{X}})-E\}(1+\Pi_++\Pi_-)|\Psi\rangle$$

$$=\begin{pmatrix} -D_{++}^2+U_{++}-P_A^2 & -i\partial_X A_{+-}-iA_{+-}\partial_X+A_{+-}^2 \\ -i\partial_X A_{-+}-iA_{-+}\partial_X+A_{-+}^2 & -D_{--}^2+U_{--}-P_-^2 \end{pmatrix}\begin{pmatrix}\chi_{++} & \chi_{+-}\\ \chi_{-+} & \chi_{--}\end{pmatrix}=0. \quad (165)$$

Here, $I$ is the identity operator.

Turning to the properties of the adiabatic basis $\{\Phi_i\}$, we must note that at each point $X$ of the base $B$ it is classified by the set of three exact quantum numbers $L=\{L,M,\xi\}$ and the three approximate ones $i=\{i_1,i_2,i_3\}$, the significance of which is established in the neighborhood of the origin $X\to 0$, and at infinity, $X\to\infty$. The rules of correspondence between the sets of asymptotic quantum numbers $i(0)$ and $i(\infty)$—the correlation diagrams—are determined by the transition operator $\mathscr{U}(0,\infty)$.

### 7.2. Reconstruction of the three-particle potential $V_{123}(\mathbf{X})$

We first consider the problem of reconstructing in the configuration space $M$ some of the previously unknown rapidly decreasing three-particle potentials $V_{123}(\mathbf{X})$. To the Faddeev equation (141) and the Schrödinger equation (142) we add certain real rapidly decreasing potentials that satisfy the conditions

$$\int d\hat{\mathbf{X}}\int_0^\infty dX\cdot X^{2\gamma+\varepsilon}|V_{123}(X,\hat{\mathbf{X}})|<\infty,\quad \varepsilon=0.1. \quad (166)$$

We shall assume that the two-body potentials $V_\alpha^n$ are known and satisfy similar conditions. We also assume that the three-particle amplitudes with potential $V_{123}(\mathbf{X})$ and without it, $f(\hat{\mathbf{X}},\mathbf{P})$ and $\mathring{f}(\hat{\mathbf{X}},\mathbf{P})$, are also known. In our case, the inverse problem reduces to finding the $S$ matrices from the amplitudes $f(\hat{\mathbf{X}},\mathbf{P})$ and $\mathring{f}(\hat{X},P)$ (37) and determining after this the effective potential and vector matrices, $U(X)$ and $A(X)$, the matrix solutions $\chi(X,P)$ of the system of equations (153), which resembles the equations of gauge theory, and, finally, the interaction potential $V_{123}(\mathbf{X})$.

First of all, we find the functions of the hyperspherical adiabatic basis $\{\Phi_j(X,\hat{\mathbf{X}})\}$ as eigenfunctions of Eq. (147) on the sphere $\hat{\mathbf{X}}$ with Hamiltonian $H^f(X,\hat{\mathbf{X}})$ specified without the potential $V_{123}(X,\hat{\mathbf{X}})$. Simultaneously, we determine the eigenvalues $\mathscr{E}_j(X)$, the terms, which depend parametrically on $X$ and give the effective potential matrix (158):

$$\mathring{U}(X)=\mathrm{diag}\,\mathscr{E}(X).$$

From the known $\{\Phi_j(X,\hat{\mathbf{X}})\}$ we find the matrix elements of the connection operator $A(X)$ (154) and, using the expressions (161), the bilocal transport operator $\mathscr{U}(X)\equiv\mathscr{U}(X,\mathring{X})$. The unitary operator $\mathscr{U}(X)$ enables us to go over from the system of equations (153) with potential matrix

$$U(X)=\mathring{U}(X)+\langle\Phi(X,\hat{\mathbf{X}}|V_{123}(X,\hat{\mathbf{X}})|\Phi(X,\hat{\mathbf{X}})\rangle \quad (167)$$

to the standard system of coupled equations (16) for the coefficients

$$\chi'(X,P)=\mathscr{U}(X)\chi(X,P) \quad (168)$$

with potential (154) in the matrix elements of the fixed frame $|e\rangle$,

$$U'(X)=\mathscr{U}(X)U(X)\mathscr{U}^{-1}(X),$$

which in the given case is conveniently chosen in the limit $\mathring{X}\to\infty$:

$$U'(X)=\mathring{U}'(X)+U'_{123}(X)$$
$$=\langle e|H^f(X,\hat{\mathbf{X}})|e\rangle+\langle e|V_{123}(X,\hat{\mathbf{X}})|e\rangle. \quad (169)$$

Since for the system of equations (16) the completeness relation (52) holds, we can formulate the inverse problem for the reconstruction of $U'_{123}(X)$ and the corresponding solutions using the generalized expressions of the multichannel inverse scattering problem (Refs. 110, 109, and 39):

$$K(X,X')+Q(X,X')+\int_{X(0)}^{\infty(X)}K(X,X'')\times Q(X'',X')dX''=0, \quad (170)$$

$$U'(X)=\mathring{U}'(X)+U'_{123}(X)=\mathring{U}'(X)\mp 2\frac{d}{dX}K(X,X), \quad (171)$$

$$\chi'(X,P)=\mathring{\chi}'(X,P)+\int_{X(0)}^{\infty(X)}K(X,X')\mathring{\chi}'(X',P)dX'. \quad (172)$$

The limits of integration in (170) and (172) and the signs in (171) depend on the particular formulation of the inverse problem. In particular, the limits of integration from $X$ to $\infty$ (respectively, from 0 to $X$) and the minus sign (respectively, plus sign) in (171) correspond to the Marchenko (respectively, Ge'fand–Levitan) method. In the $R$-matrix version of the inverse problem,[121,39] the limits of integration are $[X,a]$ and the sign is $+$, $-$ in (171). The multichannel system of equations (170) is solved for the matrix $K(X,X')$ for known $Q(X,X')$ determined by the scattering data or spectral data. The matrix of the operator of generalized shift $K(X,X')$ determines the three-particle potential matrix $U'_{123}$ and the corresponding wave functions $\chi'(X,P)$ (172) of the system of equations (16).

In the generalized approach of Marchenko, the integral kernel $Q(X,X')$ is given by

$$Q(X,X')=\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathring{F}'(X,P)(\mathring{S}-\hat{S}'(X,P))\mathring{F}'(X',P)$$

$$\times dP+\sum_{n}^{N}\mathring{F}'(X,i\kappa_n)M_n\tilde{\mathring{F}}'(X',i\kappa_n)$$

$$-\sum_{n}^{\mathring{N}}\mathring{F}'(X,i\kappa_n)\mathring{M}_n\tilde{\mathring{F}}'(X',i\mathring{\kappa}_n). \quad (173)$$

Here, the Jost solutions $\mathring{F}'_{\pm}(X,P)$ of the system of equations (16) are related to the Jost solutions $\mathring{F}_{\pm}(X,P)$ of the system (153), in accordance with the relation (168), by

$$\mathring{F}'_{\pm}(X,P)=\mathscr{U}(X,\infty)\mathring{F}_{\pm}(X,P), \quad (174)$$

and they satisfy the boundary conditions

$$\lim_{X\to\infty}\mathring{F}'_{\pm}(X,P)\exp\left[\pm i\left(XP-\frac{\pi\gamma}{2}\right)\right]$$

$$=\lim_{X\to\infty}\mathring{F}_{\pm}(X,P)\exp\left[\pm i\left(XP-\frac{\pi\gamma}{2}\right)\right]=1,$$

since, as follows from the definition (160), $\mathscr{U}(\mathring{X},\mathring{X})=1$. By virtue of the unitary freedom in the gauge of the radial functions (168), the $\hat{S}$ matrix corresponding to the system (16) is identical to the $S'$ matrix (36) for the system of equations (153); this is also true for the normalized state matrices of the discrete spectrum (18). With allowance for this the scattering matrices $\hat{S}$ and $\mathring{S}'$, which are unitary on open channels, can be determined from the known amplitudes $f(\hat{\mathbf{X}},\mathbf{P})$ and $\mathring{f}(\hat{\mathbf{X}},\mathbf{P})$ and correspond to the system of equations (16) with the three-particle potential $V_{123}$ and without it. Accordingly, $M_n$ and $\mathring{M}_n$ are real positive-definite normalization matrices of the discrete states $E_n=-(\kappa_n)^2$, $\mathring{E}_n=-(\mathring{\kappa}_n)^2$. Determining $U'_{123}$ in accordance with the expressions (170)–(172) in the representation of the fixed basis $|e\rangle$ and returning to the representation in the basis $|\Phi\rangle$, we obtain relations for the matrices of the three-particle potential $U'_{123}$ of the system (153) or (165) and the corresponding Jost solutions:

$$U_{123}(X)=U(X)-\mathring{U}'(X)$$

$$=-2\mathscr{U}-1(X,\infty)\frac{dK(X,X)}{dX}\mathscr{U}(X,\infty), \quad (175)$$

$$F_{\pm}(X,P)=\mathscr{U}^{-1}(X,\infty)F'_{\pm}(X,P). \quad (176)$$

It should be mentioned that the transport matrices $\mathscr{U}$ in (174) and (176) are the same, since they are specified on the same basis functions $|\Phi\rangle$. Finally, the physical solutions of (153) with $V_{123}$ are obtained as linear combinations of the Jost solutions $F_{\pm}(X,P)$ (41). In the generalized Gel'fand–Levitan approach, the integral kernel $Q^{\rm GL}(X,X')$ is determined from the spectral matrices $\rho(P)$ and $\mathring{\rho}(P)$ for the system (16) with the potential $V_{123}$ and without it, respectively:

$$Q^{\rm GL}(X,X')=\int_{-\infty}^{\infty}\mathring{\chi}'^{\rm reg}(X,P)d(\rho-\mathring{\rho})\tilde{\mathring{\chi}}'^{\rm reg}(X',P),$$

$$\frac{d\rho}{dE}=\frac{\rho}{\pi}|F'(P)|^{-2}\quad\text{for } E\geqslant 0, \quad (177)$$

$$\frac{d\rho}{dE}=\sum_n\delta(E-E_n)N_n'^{-1}\quad\text{for } E<0,$$

where $N_n$ are the normalization matrices determined by

$$N_n'^{-1}=\int_0^{\infty}|\chi'^{\rm reg}(X,P)|^2dX.$$

The matrix of regular solutions is distinguished by the boundary conditions

$$\lim_{X\to 0}\chi'^{\rm reg}(X,P)X^{-(K+\gamma+1)}=1.$$

After the kernel $K^{\rm GL}$ has been found from $Q^{\rm GL}$ (177) in the solution of the basic system (170) using the multichannel expressions (171), (172) of the inverse problem, we obtain the matrices of the potentials $U'_{123}$ and regular solutions $\chi'^{\rm reg}$ of the system (16). Making the inverse unitary transformation, we find $U_{123}$ and $\chi$ for the system (165).

Thus, the adiabatic representation for the three-particle wave function and the gauge transformation have made it possible to reduce correctly the three-particle inverse scattering problem for the reconstruction of $V_{123}$ to a multichannel problem and formulate it with allowance for processes involving both rearrangement of the particles and breakup.

### 7.3. Reconstruction of the three-particle potential $V_{123}$(X) and the effective potentials $\Sigma_\alpha^3 V_\alpha$(X)

We now consider the following generalization of the inverse scattering problem, in which not only the three-particle but also the two-particle potentials $V_\alpha$ ($\alpha=1,2,3$) are unknown. It reduces to the determination of effective potentials consisting of sums of two-particle potentials, $\mathring{V}(\mathbf{X})=\Sigma_\alpha V_\alpha(\mathbf{X})$, and three-particle potentials in the configuration space $\mathbf{M}$. This problem can be solved in several stages: 1) finding the $\hat{S}$ matrices from the known amplitudes $f(\hat{\mathbf{X}},\mathbf{P})$ and $\mathring{f}(\hat{\mathbf{X}},\mathbf{P})$; 2) reconstruction of the scalar potential matrix $U(X)$, the matrix of the gauge vector potential $A(X)$, and the matrix of solutions of the system of equations (153); 3) determination of the basis functions $\Phi_i(X,\hat{\mathbf{X}})$ and, finally, of the effective potentials $\mathring{V}(X,\hat{\mathbf{X}})$ and $V_{123}(X,\hat{\mathbf{X}})$ and of the complete solution $\Psi(\mathbf{X})$ in $\mathbf{M}$. As in the previous case, we use two sets of scattering data $(\hat{S}(P),\{E_n,M_n\})$, $(\mathring{\hat{S}}(P),\{\mathring{E}_n,\mathring{M}_n\})$—with the three-particle potential $V_{123}(X,\hat{\mathbf{X}})$ and without it.

First, using the scattering data $\mathring{S}(P),\{\mathring{E}_n,\mathring{M}_n\}$, we reconstruct the effective potential matrix $\mathring{U}'(X)$:

$$\mathring{U}'(X)=\mathscr{U}(X)\mathscr{E}(X)\mathscr{U}^{-1}(X)$$

and we find the solutions of the system (16), using the ordinary or generalized Gel'fand–Levitan–Marchenko expressions (170)–(172). We can then find the transport operator $\mathscr{U}(X)$ and the energy $\mathscr{E}(X)$ by solving the eigenvalue problem

$$\mathring{U}'(X)\mathscr{U}(X)=\mathscr{U}(X)\mathscr{E}(X). \quad (178)$$

Thus, the terms $\mathscr{E}(X)$ are determined by solving the inverse problem for the system of equations (16) and the subsequent diagonalization procedure (178), whereas in the previous case we solved the direct eigenvalue problem for the reference equation (147). Moreover, knowledge of $\mathscr{U}(X)$ also enables us to reconstruct the matrix elements of the effective vector potential,

$$A(X)=\mathscr{U}^{-1}(X)\frac{d}{dX}\mathscr{U}(X)+\gamma X^{-1}, \tag{179}$$

which is responsible for the velocity-dependent potential $A(X)/(d/dX)$ in (153). However, having reconstructed the potential matrix $\mathring{U}(X)$, we still cannot yet determine in the usual manner the effective potential

$$\mathring{V}(X,\hat{\mathbf{X}})=\sum_{\alpha}V_{\alpha}(X,\hat{\mathbf{X}})$$
$$=\sum_{ij}\Phi_i(X,\hat{\mathbf{X}})\mathring{U}_{ij}(X)\Phi_j^*(X,\hat{\mathbf{X}}), \tag{180}$$

since we do not know the basis functions $\Phi_i(X,\hat{\mathbf{X}})$, which are determined by the same unknown potential $\mathring{V}(X,\hat{\mathbf{X}})$ in the solution of (147) for each fixed value of the "slow" variable $X$. It is now necessary to solve a parametric inverse Sturm–Liouville problem on the basis of the technique developed in the previous section.[14] The input data for such a problem are the spectral characteristics $\{\mathscr{E}_j(X),\gamma_j(X)\}$, the terms and normalization constants, which depend parametrically on $X$. The terms are determined from (178), and the normalization constants in accordance with the relation (94),

$$\gamma_j^{-1}(X)=\sum_j\mathscr{U}_{ji}(X)\gamma_i^{-1}\mathscr{U}_{ij}(X), \tag{181}$$

where $\gamma_i^{-1}$ is the usual normalization:

$$\gamma_i^{-1}=\int_0^{\infty}|e_i(\hat{\mathbf{X}})|^2d\hat{\mathbf{M}}^{\gamma}.$$

Then, solving the parametric Sturm-Liouville problem,[14] we reconstruct the effective potential $\mathring{V}(X,\hat{\mathbf{X}})=\Sigma_{\alpha}V_{\alpha}(X,\hat{\mathbf{X}})$ and the corresponding solutions $\Phi_i(X,\hat{\mathbf{X}})$.

Finally, in the second stage, using the scheme (170)–(175) described above, we reconstruct the three-particle potential matrix $U_{123}(X)$ and the corresponding solutions from the two sets of scattering data $(\mathring{S}(P),\{\mathring{E}_n,\mathring{M}_n\})$ and $(\hat{S}(P),\{E_n,M_n\})$.

The existence of the "global" adiabatic basis $\{\Phi_i\}$, and also the possibility of determining the effective scalar, $\mathscr{U}(X)$, $U_{123}(X)$, and vector, $A(X)$, potentials from the data of the three-particle scattering problem are due to the fact that in our approach information on the interaction of the fragments is contained not only in the radial solutions $\chi$ but also in the basis "quasiangular" functions $\Phi_i$, in contrast to the traditional approaches using expansions with respect to $K$ harmonics or cluster functions. Moreover, there is a possibility to use the technique of Bargmann potentials for the analytic modeling of effective three-particle interactions and finding corresponding exact solutions not only for the systems of coupled equations but also for the parametric equation (Sec. 6).

## 8. SUPERSYMMETRY OF GAUGE EQUATIONS

Thus, we have seen that induced gauge potentials arise naturally in the descriptions of quantum-mechanical systems that depend both on slowly varying external parameters and on fast internal variables. This occurs in many real systems in which there are slow and fast degrees of freedom and it is necessary to estimate the influence of the slow dynamics on the fast and vice versa. Accordingly, the total Hamiltonian decomposes into two components: $H=H^f+H^s$, where $H^f(\mathbf{R})$ is a parametric family of fast Hamiltonians. The wave function of the total Hamltonian $H$ can be represented as an expansion with respect to the eigenfunctions $\Phi_n(\mathbf{R};\mathbf{r})$ of the instantaneous Hamiltonian $H^f$ for each fixed value of the slow variables $\mathbf{R}$ (3):

$$\Psi(\mathbf{R},\mathbf{r})=\sum_n\int\Phi_n(\mathbf{R};\mathbf{r})F_n(\mathbf{R}),$$
$$H^f(\mathbf{R})\Phi_n(\mathbf{R};\mathbf{r})=E_n(\mathbf{R})\Phi_n(\mathbf{R};\mathbf{r}). \tag{182}$$

Substituting the expansion for $\Psi$ in the original Schrödinger equation (5), $H\Psi(\mathbf{R},\mathbf{r})=E\Psi(\mathbf{R},\mathbf{r})$, and using the orthogonality relation $\langle n|m\rangle=\delta_{nm}$ and the completeness relation $|n\rangle\langle n|=\mathbf{1}\delta(\mathbf{r}-\mathbf{r}')$ for the eigenfunctions $|\Phi_n(\mathbf{R};\mathbf{r})\rangle$ of the Hamiltonian $H^f(\mathbf{R})$ for fixed $\mathbf{R}$, we obtain a multidimensional system of equations of gauge type (6) for the expansion coefficients $F_n$:

$$-1/2[I\nabla-i\mathbf{A}(\mathbf{R})]^2F(\mathbf{R})+V(\mathbf{R})F(\mathbf{R})=EF(\mathbf{R}), \tag{183}$$

where $F=\{F_n\}$ is a column vector of dimension $M$, $I$ is the unit matrix, and $A$ and $V$ are the vector and scalar components of the gauge field:

$$A_{nm}(\mathbf{R})=\langle\Phi_n|i\nabla|\Phi_m\rangle, \tag{184}$$

$$V_{nm}(\mathbf{R})=\langle\Phi_n|H^f|\Phi_m\rangle\delta_{nm}+1/2\sum_{k\neq n,m}A_{nk}A_{km}$$
$$+\langle\Phi_n|V^s|\Phi_m\rangle. \tag{185}$$

Equation (183) possesses the unitary gauge symmetry of the $U(M)$ group. For the complete set $\Phi_n$, the second term in (185) vanishes, and if $A$ does not have singularities, one can find a gauge (pure gauge) in which the induced vector potential vanishes. In the single-level approximation, $F(R)$ becomes a scalar wave function.

We first consider a one-dimensional problem. We represent the Hamiltonian $H$ of Eq. (183) in the factorized form

$$H^-=1/2Q^-Q^+,\quad H^+=1/2Q^+Q^- \tag{186}$$

with

$$Q^{\pm}=\pm D+\alpha(R),\quad D=\partial_R-iA(R). \tag{187}$$

It follows from this that

$$H^{\pm}=1/2\{[-D^2+\alpha^2(R)]I+\sigma_3(D\alpha(R)-\alpha(R)D)\},$$
$$V_+(R)=V_-(R)+D\alpha(R)-\alpha(R)D. \tag{188}$$

Here

$$\sigma_3=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}.$$

In two-component notation, the supersymmetric Hamiltonian (188) can be rewritten in the form

$$H^s=1/2\{Q_-,Q_+\}=\begin{pmatrix}H^+ & 0\\ 0 & H^-\end{pmatrix},\qquad(189)$$

where $Q_-$ and $Q_+$ are defined as ordinary supersymmetric charges:

$$Q_+=\begin{pmatrix}0 & Q^+\\ 0 & 0\end{pmatrix},\quad Q_-=\begin{pmatrix}0 & 0\\ Q^- & 0\end{pmatrix},\qquad(190)$$

and they satisfy the relations of the supersymmetry algebra

$$Q_+^2=Q_-^2=[H^s,Q_+]=[H^s,Q_-]=0.\qquad(191)$$

If we ignore the nondiagonal transition elements in the system of equations (183), we obtain a system of uncoupled equations, i.e., the adiabatic approximation. In this case, the supersymmetric Hamiltonian (189) can be rewritten in the form

$$H^s=1/2\{[-D^2+\alpha^2(R)]I+\sigma_3\partial_R\alpha(R)\},\qquad(192)$$

where $\alpha(R)$ is related to $V(R)$, as usual, by the Riccati equation: $\alpha^2(R)-\partial_R\alpha(R)=V_-(R)$. In the $\alpha(R)=\partial_R W$ representation of the ground-state wave function,[124] the wave functions of $H^+$ and $H^-$ for arbitrary energy are related by the transformations

$$\chi_+(R,E)=Q^+\chi_-(R,E)=[D+\alpha(R)]\chi_-(R,E)$$
$$=\chi_0^{-1}(R)W_D\{\chi_0(R),\chi_-(R,E)\},\qquad(193)$$

which generalize the Darboux–Crum–Kreĭn transformations. Here, $\chi_0(R)$ is the ground-state wave function of $H^-$, represented in the form

$$\chi_0(R)=P\exp\left\{-i\int^R A(R')dR'\right\}\exp\left\{-\int^R\alpha(R')dR'\right\},\qquad(194)$$

the generalized Wronskian $W_D$ is defined in terms of the covariant derivative,

$$W_D=\{\chi_0^\dagger D\chi_- -(D\chi_0)^\dagger\chi_-\},$$

and $P$ denotes the ordered exponential.

Further, following Refs. 31, 25, and 26, we can represent the three- or two-dimensional generalization of (186)–(194) for the systems of gauge equations (183), which describe the slow dynamics of nonrelativistic systems (in particular, the three-particle system) and the supersymmetry for them. We introduce supercharges as follows:

$$\bar{\mathbf{Q}}^+=\frac{1}{\sqrt{2}}\tau_+\sigma\mathbf{Q}^+=\frac{1}{\sqrt{2}}\tau_+\sum_\mu\sigma_\mu Q_\mu^+,$$
$$\bar{\mathbf{Q}}^-=\frac{1}{\sqrt{2}}\tau_-\sigma\mathbf{Q}^-=\frac{1}{\sqrt{2}}\tau_-\sum_\mu\sigma_\mu Q_\mu^-,\qquad(195)$$

where $\bar{\mathbf{Q}}^+$ and $\bar{\mathbf{Q}}^-$ are $2\times2$ block matrices formed, like (191), from the multidimensional multichannel supersymmetry generators $\mathbf{Q}^+$ and $\mathbf{Q}^-$, the coordinate components of which are defined in accordance with (187):

$$Q_\mu^\mp=\pm D_\mu+\partial_\mu W,\quad D_\mu=\partial_\mu-iA_\mu,\qquad(196)$$

$\tau_\pm=1/2(\sigma_1\pm i\sigma_2)$, and $\sigma_\mu$ ($\mu=1, 2, 3$) are the Pauli spin matrices. Then the supersymmetric Hamiltonian can be represented in the form

$$H^s=\frac{1}{2}\begin{pmatrix}(\sigma\mathbf{Q}^+)(\sigma\mathbf{Q}^-) & 0\\ 0 & (\sigma\mathbf{Q}^-)(\sigma\mathbf{Q}^+)\end{pmatrix}=\frac{1}{2}\begin{pmatrix}(\mathbf{Q}^+\mathbf{Q}^-)+i\sigma(\mathbf{Q}^+\times\mathbf{Q}^-) & 0\\ 0 & (\mathbf{Q}^-\mathbf{Q}^+)+i\sigma(\mathbf{Q}^-\times\mathbf{Q}^+)\end{pmatrix}.\qquad(197)$$

Besides the ordinary components

$$\frac{1}{2}(\mathbf{Q}^+\mathbf{Q}^-)=\frac{1}{2}\sum_\mu Q_\mu^+Q_\mu^-,\quad\frac{1}{2}(\mathbf{Q}^-\mathbf{Q}^+)=\frac{1}{2}\sum_\mu Q_\mu^-Q_\mu^+,$$

which correspond to the sum of one-dimensional supersymmetric Hamiltonians (186), interesting new terms arise in $H^s$ (197):

$$i\sigma[\mathbf{Q}^+\times\mathbf{Q}^-]=i\sigma B-2\sigma[\nabla W\times\pi]$$
$$=i\sigma F-2\sigma(\nabla W\times\mathbf{p})+2e\sigma(\nabla W\times\mathbf{A}),\qquad(198)$$

where $\pi=\mathbf{p}-e\mathbf{A}$, the matrix tensor

$$F_{\mu\nu}=\partial_\mu A^\nu-\partial_\nu A^\mu-i[A^\nu,A^\mu]=\varepsilon^{\nu\mu k}B^k\qquad(199)$$

is identified with the magnetic field, the term $-2\sigma(\nabla W\times\mathbf{p})$ goes over for central fields into the spin–orbit coupling term $(2/R)\partial_R W(\sigma\cdot\mathbf{L})$ (Refs. 31 and 32), and, finally, the new term $2\sigma(\nabla W\times\mathbf{A})$, which was omitted in Ref. 25, realizes a coupling between the vector potential and the gradient of the scalar potential. Its role will become completely clear in what follows in the investigation of geometric phases.

The procedure (186)–(197) generalizes the usual procedure of supersymmetric nonrelativistic quantum mechanics and is based on factorization of the total Hamiltonian of Eq. (183). It enables one to generate a large class of exactly solvable three-particle and multidimensional models in addition to those of Refs. 4 and 5 for Eqs. (183) in the presence of effective vector potentials.

We analyze the approach in more detail, directly generalizing the well-known problem of the motion of a charged particle with spin 1/2 under the influence of an inhomogeneous magnetic field having direction perpendicular to the plane of motion of the particle.[29,34] In this case, the supersymmetry is based on the fact that in two-dimensional space the Pauli Hamiltonian

$$H^P=1/2[-i\nabla-e\mathbf{A}]^2-(e/2)\sigma_3\mathbf{B} \tag{200}$$

can be expressed as the square of the Dirac Hamiltonian ($h=c=m=1$):

$$H^P=1/2(H^D)^2,$$
$$H^D=(\sigma\cdot\pi)=\sigma(\mathbf{p}-e\mathbf{A}). \tag{201}$$

We define the Hermitian supercharges $Q_i=Q_i(x,y)$:

$$Q_1=1/2[\sigma_1(\pi_x-\partial_yW)+\sigma_2(\pi_y+\partial_xW)],$$
$$Q_2=1/2[\sigma_2(\pi_x-\partial_yW)-\sigma_1(\pi_y+\partial_xW)] \tag{202}$$

with additional scalar functions $\partial_\mu W$ compared with the classical problem (200)–(201). This is a direct generalization of the definition of supercharges introduced by Witten[30] to describe the motion of a nonrelativistic particle with spin 1/2 in one-dimensional space:

$$Q_1=1/2[\sigma_1p_x+\sigma_2W(x)],$$
$$Q_2=1/2[\sigma_2p_x-\sigma_1W(x)]. \tag{203}$$

In the multichannel case of the adiabatic approach, $W(x,y)$ and $\pi_\mu$ are matrices; $\pi_\mu$ are related to the covariant derivative $D$ by

$$\pi_\mu=-iD_\mu=[-iI\partial_\mu-A_\mu],$$

where the matrix $A_\mu$ is determined from (184). The supercharges $Q_i$ satisfy the set of relations of Witten's supersymmetric quantum mechanics:

$$\{Q_i,Q_j\}=\delta_{ij}H \quad \text{and} \quad [H,Q_i]=0 \quad (i=1,...,N). \tag{204}$$

In the considered case, $i=2$. As usual, we introduce the non-Hermitian supercharges

$$\bar{\mathbf{Q}}^+=\frac{1}{\sqrt{2}}[-Q_2+iQ_1], \quad \bar{\mathbf{Q}}^-=\frac{1}{\sqrt{2}}[-Q_2-iQ_1]. \tag{205}$$

We represent the matrix supercharges $\bar{\mathbf{Q}}^+$ and $\bar{\mathbf{Q}}^-$ as $2\times2$ block matrices,

$$\bar{\mathbf{Q}}^+=\frac{1}{\sqrt{2}}\tau_+[(i\pi_x+\partial_xW)+(\pi_y-i\partial_yW)]$$
$$=\frac{1}{\sqrt{2}}\tau_+[Q_x^+-iQ_y^+],$$
$$\bar{\mathbf{Q}}^-=\frac{1}{\sqrt{2}}\tau_-[(-i\pi_x+\partial_xW)+(\pi_y+i\partial_yW)]$$
$$=\frac{1}{\sqrt{2}}\tau_-[Q_x^-+iQ_y^-], \tag{206}$$

which correspond to (195). Using them, we construct the supersymmetric Hamiltonian

$$H^s=1/2\{\bar{\mathbf{Q}}^+,\bar{\mathbf{Q}}^-\}=\frac{1}{2}\begin{pmatrix}(\mathbf{Q}^+\cdot\mathbf{Q}^-)+i(\mathbf{Q}^+\times\mathbf{Q}^-) & 0\\ 0 & (\mathbf{Q}^-\cdot\mathbf{Q}^+)-i(\mathbf{Q}^-\times\mathbf{Q}^+)\end{pmatrix}. \tag{207}$$

The supercharges $\bar{\mathbf{Q}}^+$ and $\bar{\mathbf{Q}}^-$ satisfy the relations (191) of the supersymmetry algebra.

The superpartners $H^+$ and $H^-$ of the Hamiltonian $H^s$ (207) can be represented as follows:

$$H^+=1/2[Q_x^+Q_x^-+Q_y^+Q_y^-+i(Q_x^+Q_y^--Q_y^+Q_x^-)],$$
$$H^-=1/2[Q_x^-Q_x^++Q_y^-Q_y^+-i(Q_x^-Q_y^+-Q_y^-Q_x^+)]. \tag{208}$$

Using the definitions (206) and making simple manipulations, we can write the Hamiltonians $H^\pm$ in the explicit form

$$H^+=1/2[\pi_x^2+\pi_y^2+i(\pi_x\pi_y-\pi_y\pi_x)+(\partial_xW)^2+(\partial_yW)^2$$
$$+(i\pi_x+\pi_y)(\partial_xW+i\partial_yW)+\partial_xW\pi_y-\partial_yW\pi_x], \tag{209}$$
$$H^-=1/2[\pi_x^2+\pi_y^2-i(\pi_x\pi_y-\pi_y\pi_x)+(\partial_xW)^2+(\partial_yW)^2$$
$$+(-i\pi_x+\pi_y)(\partial_xW-i\partial_yW)+\partial_xW\pi_y-\partial_yW\pi_x].$$

We rewrite these relations in a form more convenient for analysis:

$$H^+=\tfrac{1}{2}[\pi^+\pi^-+(\partial_xW)^2+(\partial_yW)^2$$
$$+\pi^+(\partial_xW-i\partial_yW)+\partial_xW\pi_y-\partial_yW\pi_x],$$
$$H^-=\tfrac{1}{2}[\pi^-\pi^++(\partial_xW)^2+(\partial_yW)^2$$
$$+\pi^-(\partial_xW+i\partial_yW)+\partial_xW\pi_y-\partial_yW\pi_x]. \tag{210}$$

We have here taken into account the usual definition of non-Hermitian supercharges,

$$\pi^+=(+i\pi_x+\pi_y), \quad \pi^-=(-i\pi_x+\pi_y),$$

which generate the matrix analog of the Pauli Hamiltonian (200), written in the form

$$H^P=1/2\{\pi^+,\pi^-\} \tag{211}$$

with scalar potential matrix $V=\sigma_3F_{xy}$. As follows from the relations (206), for $W(x,y)=0$ the non-Hermitian supercharges $\mathbf{Q}^+$ and $\mathbf{Q}^-$ go over into $\pi^+$ and $\pi^-$, and the supersymmetric Hamiltonian $H^s$ (207) goes over into $H^P$. This is equivalent to the principle of minimal coupling to the electromagnetic field as with gauge $A$. If the vector potential is $A_\mu=0$, then from the relations (202)–(207) we readily obtain the Witten construction[30] of supersymmetric quantum mechanics in two-dimensional space.[31,32] This becomes obvious if the generators $i\pi_x$ and $i\pi_y$ are replaced by ordinary partial derivatives in Eqs. (209) and (210). In the general case, replacing $\pi$ by $\mathbf{p}$ for Eqs. (195)–(197), we obtain for the Schrödinger equation the supersymmetric Hamiltonian

$$\hat{H}=\frac{1}{2}\begin{pmatrix}(\sigma\cdot\mathbf{Q}^+)(\sigma\cdot\mathbf{Q}^-) & 0\\ 0 & (\sigma\cdot\mathbf{Q}^-)(\sigma\cdot\mathbf{Q}^+)\end{pmatrix}=\frac{1}{2}\begin{pmatrix}(\mathbf{Q}^+\cdot\mathbf{Q}^-)-2\sigma(\nabla W\times\mathbf{p}) & 0\\ 0 & (\mathbf{Q}^-\cdot\mathbf{Q}^+)+2\sigma(\nabla W\times\mathbf{p})\end{pmatrix}. \quad (212)$$

We now redefine the coordinate components $\mathbf{Q}^\pm$:$Q_\mu^\pm=\pm\partial_\mu+\partial_\mu W$ and use $\mathbf{p}=-i\nabla$.

The relations (200)–(207) can be generalized to the 4-component case in two ways. One of them corresponds to the choice of the 4-component supercharges determined by the relation (195) with $\mu=1$ and 2. The two-dimensional supersymmetric Hamiltonian is determined from the relation (197), which directly generalizes (207):

$$H^s=\frac{1}{2}\begin{pmatrix}(\mathbf{Q}^+\cdot\mathbf{Q}^-)+i\sigma_3(\mathbf{Q}^+\times\mathbf{Q}^-) & 0\\ 0 & (\mathbf{Q}^-\cdot\mathbf{Q}^+)+i\sigma_3(\mathbf{Q}^-\times\mathbf{Q}^+)\end{pmatrix}. \quad (213)$$

The second method follows the logic of choosing the supercharges $Q_1$ and $Q_2$ determined by the relation (202), where $\sigma_\mu$ in the expression for the supercharge $Q_1$ must be replaced by $\alpha$ matrices. In this case, it is more convenient to introduce in place of the supercharges $Q_\mu^\pm$ the supercharges

$$\Pi_x^\pm=\pm D_x-i\partial_y W,\quad \Pi_y^\pm=\pm D_y+i\partial_x W. \quad (214)$$

Then $\bar{\mathbf{Q}}^+$ and $\bar{\mathbf{Q}}^-$ (195) are redefined as follows:

$$\bar{\Pi}^+=\frac{1}{\sqrt{2}}\tau_+\sum_\mu\sigma_\mu\Pi_\mu^+,\quad \bar{\Pi}^-=\frac{1}{\sqrt{2}}\tau_-\sum_\mu\sigma_\mu\Pi_\mu^-, \quad (215)$$

and in place of the supersymmetric Hamiltonian (213) we have

$$H^s=\frac{1}{2}\begin{pmatrix}(\Pi^+\Pi^-)+i\sigma_3(\Pi^+\times\Pi^-) & 0\\ 0 & (\Pi^-\Pi^+)+i\sigma_3(\Pi^-\times\Pi^+)\end{pmatrix}. \quad (216)$$

Do our relations (216) and (213) satisfy the principle of minimum coupling of gauge fields? If we represent the gauge vector potential in the relations (202) and (216) in the form

$$A_x\to A_x+\partial_y W;\quad A_y\to A_y-\partial_x W$$

(we emphasize that this is not a gauge transformation), then the matrix tensor $B_{xy}$ can be represented as

$$B_{xy}=\partial_x(A_y-\partial_x W)-\partial_y(A_x+\partial_y W), \quad (217)$$

and $H^s$ (216) with a noncentral field can be expressed in the form of the Pauli equation (200) with $H^P$ if we make the substitutions

$$\pi^+\to\bar{\Pi}^+,\quad \pi^-\to\bar{\Pi}^-.$$

For the definition of the supercharges by the relation (195), the supersymmetric Hamiltonian (197) is identical to the Pauli equation if we make the substitution

$$A_\mu\to A_\mu+i\partial_\mu W.$$

This is none other than a gauge transformation of the two superpartners of the Pauli Hamiltonian with $W=0$, i.e., for the relation (213) the principle of minimum coupling of gauge fields is satisfied. After this, it is obvious that the supersymmetric Hamiltonian of the Schrödinger equation (212) can be represented as the Pauli operator (200) if in Eq. (212) we replace $i\partial_\mu W$ by $A_\mu$.

We investigate the relations (202), (214), and (216), which are a natural generalization of the Witten construction for systems of gauge equations in two-dimensional space. The three-dimensional model of supersymmetric quantum mechanics for systems of gauge equations is obtained similarly.

If such an approach is applied to charged particles with spin 1/2, the so-called spin-flip effect holds for simultaneous change of the coordinate dependence of the wave functions, when the generators $(\sigma\cdot\Pi^+)$ and $(\sigma\cdot\Pi^-)$ carry the superpartner states into each other:

$$\chi_+=(\sigma\cdot\Pi^+)\chi_-\quad\text{and}\quad\chi_-=(\sigma\cdot\Pi^-)\chi_+.$$

If $\chi_{-\sigma_3}$ is an $H^-$ eigenstate of the Hamiltonian (216), then the $H^+$ eigenstate is

$$\chi_{+\sigma_3}=i[(\sigma_1(\pi_x-\partial_y W)+\sigma_2(\pi_y+\partial_x W)]\chi_{-\sigma_3}. \quad (218)$$

The zeroth eigenstate $\chi_0=\chi_{-\sigma_3}$ of the Hamiltonian $H^-$ must be annihilated by $(\sigma\cdot\Pi^+)$:

$$(\sigma\cdot\Pi^+)\chi_0=0. \quad (219)$$

We multiply this relation from the left by $\sigma_2$, obtaining

$$\{\sigma_3[\partial_x - i(A_x+\partial_y W)] + (i\partial_y + A_y + i\partial_x W)\}\chi_0 = 0. \tag{220}$$

Recalling now that for a divergenceless vector potential

$$A_x = -\partial_y\Phi, \quad A_y = -\partial_x\Phi,$$

we seek the solution of Eq. (220) in the form

$$\chi_0(x,y) = f(x,y)\exp\{-\sigma_3(\Phi - W)\}, \tag{221}$$

where $\Phi = \int\int F_{xy}(x,y)dxdy$ is defined as the ordinary flux, and $F_{xy}(x,y) = \partial_x A_y - \partial_y A_x - i[A_x, A_y]$.

It is now easy to see that in the adiabatic representation (183) we arrive at the well-known situation established in Ref. 34, namely, the ground state of the Hamiltonian $H^-$ (207) and (216) is degenerate, and the number of zero modes is determined by the Atiyah–Singer index theorem:

$$\chi_0(x,y) = f(x,y)P\exp\left\{-\sigma_3\int\int B_{xy}(x,y)dxdy\right\}, \tag{222}$$

where the tensor $B_{xy}$ is expressed by the relation (217) and the function $f(x,y)$ satisfies the equation

$$(\partial_x + i\sigma_3\partial_y)f(x,y) = 0.$$

Therefore, for the supersymmetric Hamiltonian (216) with scalar potential $W(x,y) \neq 0$ the function $f(x,y)$ is an entire function of $(x + i\sigma_3 y)$, as was shown by Aharonov and Casher[34] for the case with zero scalar potential, $W(x,y) = 0$. Then as in the case $W(x,y) = 0$, we have degeneracy of the ground state. The multiplicity of the degeneracy of the zero modes for a particle moving in an external gauge field is related to the topological number $N$, which is defined as a surface integral of the matrix tensor $B_{xy}(x,y)$:

$$\int\int B_{xy}(x,y)dxdy = \Phi - W = 2\pi(N+\varepsilon), \quad 0 < \varepsilon < 1. \tag{223}$$

Here, $N$ determines the multiplicity of the degeneracy of the zero-energy eigenstates. If $\varepsilon = 0$, the flux, defined as the difference of the ordinary flux and the additional scalar potential, is quantized, and one can speak of a quantum Hall effect and nonstandard statistics in diatomic systems.

It is now readily seen from (222) and (217) that the presence of a scalar potential may lead to an increase and, consversely, to a decrease or even cancellation of the positive index $N$; i.e., it may influence the existence of the quantum Hall effect and may lead to spontaneous supersymmetry breaking. The presence or absence of degeneracy resulting from supersymmetry does not rule out other geometric phases, in particular Berry phases[15] or nonadiabatic Aharanov–Anandan phases.[17,36]

## 9. GEOMETRIC PHASES

Cases of crossing of potential curves, which lead to singularities of the gauge vector potential $A$, are particularly interesting. In the presence of singularities of $A$, there are nontrivial geometric phases in the wave functions $\chi_0$, and, therefore, in $\chi(R,E)$ determined for arbitrary energies (193):

$$\delta = \frac{1}{2} i \oint_C A(R)dR. \tag{224}$$

These phases must be taken into account in addition to the already determined geometric phases.

We consider some simple examples of the one-dimensional problem. For $\alpha(R) = 0$ (or, which is the same thing, $W = 0$) the two supersymmetric partners of the one-dimensional problem are, as follows from (188), identical, in contrast to the $N$-dimensional, $N \geqslant 2$, problem, i.e., the supersymmetry in this case is determined by the scalar potential $\alpha(R)$.

Let $\alpha(R) = -iA(R)$. Then

$$\begin{aligned} V^+(R) &= 1/2[-A^2 - i\partial_R A], \\ V^-(R) &= 1/2[-A^2 + i\partial_R A]. \end{aligned} \tag{225}$$

For convenience, we have adopted the notation $\partial_R = d/dR$. With allowance for (225), Eq. (183) can be rewritten in the form

$$-1/2(\partial_R - iA(R))^2\chi + 1/2[-A^2 \mp i\partial_R A(R)]\chi = E\chi, \tag{226}$$

which corresponds to the equations

$$\begin{aligned} &-1/2\partial_R^2\chi + iA(R)\partial_R\chi = E\chi, \\ &-1/2\partial_R^2\chi + i(\partial_R A(R) + A(R)\partial_R)\chi = E\chi, \end{aligned} \tag{227}$$

which are represented in the factorized form (186) with supercharges determined in accordance with the expressions (187):

$$\begin{aligned} Q^- &= -\partial_R + iA(R) + \alpha(R) = -\partial_R, \\ Q^+ &= +\partial_R - iA(R) + \alpha(R) = \partial_R - 2iA(R). \end{aligned} \tag{228}$$

In the single-channel case, $A(R)$ is purely imaginary. If $\chi_0(R)$ is the ground state (227) of the Hamiltonian $H^-$, then $Q^+(R)$ must annihilate it:

$$Q^+\chi_0 = 0 \text{ or } (\partial_R - 2iA(R))\chi_0(R) = 0.$$

From this we have

$$\begin{aligned} \chi_0(R) &= P\exp\left(2i\int_{\mathring{R}}^R A(R')dR'\right) \\ &= P\exp\left(-2\int_{\mathring{R}}^R \alpha(R')dR'\right). \end{aligned} \tag{229}$$

Recalling the definition of the gauge transformation

$$U(R,\mathring{R}) = P\exp\left(\int_{\mathring{R}}^R A(R')dR'\right),$$

we see that we have obtained a simple example of an exactly solvable model in which the gauge transformation is, apart from a coefficient, identical to the ground-state function $\chi_0(R)$.

In the study of free motion in $N$-dimensional space in a spherical parametrization $(\mathbf{R}^1 \times \mathbf{S}^{N-1})$, it is necessary to introduce a geometric phase in order to take into account the defect of the parametrization of space by spherical coordinates and the resulting appearance of a singular point at the origin. To Eq. (226) there corresponds $A(R)=i\nu R^{-1}$ [$\nu=(N-1)/2$], when in accordance with (225)

$$V^{+}(R)=\nu(\nu-1)/R^2, \quad V^{-}(R)=\nu(\nu+1)/R^2.$$

The phases are defined in accordance with (224):

$$\delta_{+}=(\pi\nu)/2, \quad \delta_{-}=\pi(\nu+1)/2.$$

In the three-particle problem considered for a hyperspherical parametrization of space, there arises at the point of triple collision an analogous singular potential and an associated phase, these corresponding to $\nu=5/2$. In general, the singularities of $A(R)$ are not necessarily at the origin $R=0$, and the functional dependence $A(R)$ may be more complicated. For example, if

$$A(R)=i\prod_j f(R)/(R-ib_j) \tag{230}$$

with a smooth function $f(R)$ that has an analytic continuation into the complex plane of $R$, then

$$\delta=\frac{\pi i}{2}\sum_j f(ib_j). \tag{231}$$

For $f(R)=2R$ or $f(R)=(R+ib_j)$, the geometric phase is determined as follows: $\delta=\pi\Sigma_j b_j$. In general, $R$ must be scaleless and dimensionless. It can then be seen that $\delta$ for even values of $b$ does not change sign, but it does for odd values, i.e., this is a kind of analog of the Aharanov–Bohm effect.

We return to the more complicated problem associated with Eqs. (183) and (184). The nondiagonal elements $A_{nm}(R)=\langle n|i\nabla_R|m\rangle$ realize transitions between different eigenstates of the fast Hamiltonian $H^f$. Here, the approximation of adiabaticity does not apply, in particular, in the neighborhood of a point of term crossing. We rewrite the matrix elements of the induced vector potential (184) in the different form

$$A_{nm}(R)=i\frac{\langle\Phi_n(R;r)|\partial_R H^f(R)|\Phi_m(R;r)\rangle}{E_n(R)-E_m(R)}, \tag{232}$$

which is obtained by differentiating Eq. (182) with respect to $R$ and using the orthonormality relations for the basis functions $\Phi$.

It is obvious that when the terms $E_n(R)$ cross or quasicross at certain points $R=R_m$ singularities responsible for geometric phases arise in the matrix elements $A_{nm}(R)$. It is convenient to introduce the matrix of geometric phase factors

$$S_{nm}=\exp\,\mathrm{Im}\left(\oint A_{nm}(R)dR\right). \tag{233}$$

In our case, we obtain

$$S_{nm}=\exp\left(\pi i\sum_m \mathrm{Res}\frac{\langle\Phi_n(R;r)|\partial_R V(R,r)|\Phi_m(R;r)\rangle}{E_n(R)-E_m(R)}\right). \tag{234}$$

It is easy to write down the generalization to the multichannel case of the single-channel supersymmetric model (225)–(229):

$$\chi_0(R)=CU(R,\mathring{R})=C\exp\left(i\int_{\mathring{R}}^{R}A(R')dR'\right)$$

with $A_{nm}$ determined by the relation (232). It follows from this that all information about the term crossing is contained in the ground-state function.

If an incomplete set $\Phi$ is taken into account, the state vector is defined on an $n$-dimensional subspace of the $M$-dimensional Hilbert space [$M=(n+m)$], the second term in (185) does not vanish, and nontrivial gauge fields are induced. Then we have the Berry phases

$$\delta_{ii}=\sum_{j\neq i}^{n}\oint A_{ij}A_{ji} = \sum_{j\neq i}^{n}\oint\langle\Phi_i|\partial_R|\Phi_j\rangle\langle\Phi_j|\partial_R|\Phi_i\rangle \tag{235}$$

as in Ref. 15. There also appear geometric phases associated with nondiagonal elements of the effective matrix of vector potentials $A_{ik}(R)=A_{ij}(R)A_{jk}(R)$. In this case, it is better to use the geometric $S$ matrix (233):

$$S_{ik}=\exp\left(\pi i\sum_{j\neq i,k}^{n}\mathrm{Res}\frac{\langle\Phi_i(R;r)|\partial_R H^f(R;r)|\Phi_j(R;r)\rangle}{E_i(R)-E_j(R)} \times\frac{\langle\Phi_j(R;r)|\partial_R H^f(R;r)|\Phi_k(R;r)\rangle}{E_j(R)-E_k(R)}\right). \tag{236}$$

In this last case, there may even be crossing of three terms at one point. Thus, we see that a non-Abelian nonadiabatic phase is manifested even when there is only a radial dependence. The reason for this is the presence of singularities in $A(R)$. The gauge transformation $U(R,\mathring{R})=\int_{\mathring{R}}A(R')dR'$ does not eliminate them, and they appear in the scalar potential in the same way as the nonvanishing curl of the vector potential in the $N$-dimensional space of the slow variables.

On the reduction of the multichannel system of gauge equations to a finite system, one can obtain a system of uncoupled effective equations[125] of the form

$$\left\{\mu(R)\frac{d}{dR}\mu^{-1}(R)\frac{d}{dR}+\mu(R)(P^2-\bar{V}(R))\right\}\Psi(R)=0 \tag{237}$$

with nonzero diagonal connection

$$\bar{A}=\mathrm{diag}\,\bar{A}(R)=1/2\mu(R)\frac{d}{dR}\mu^{-1}(R),$$

where

$$\mu^{-1}(R)=1-(2M)^{-1}\sum_j^N\frac{A_{ij}(R)A_{ji}(R)}{E_i(R)-E_j(R)}. \tag{238}$$

The applicability of the expressions (235) and (236) is here obvious.

In the absence of the singular behavior $\mu^{-1}(R)$, as holds far from term crossing, Eqs. (237) can be represented by means of a Liouville transformation in the form (1), and for them, as for the Schrödinger equation, one can construct exactly solvable models and, therefore, obtain solutions in closed analytic form.

## 10. GENERALIZED ALGEBRAIC BARGMANN–DARBOUX TRANSFORMATIONS

We realize algebraic Darboux–Crum–Kreĭn and Bargmann transformations for equations of the more general form (1) than the Schrödinger equation:

$$-\frac{d^2\psi(r)}{dr^2}+V(r)\psi(r)+\frac{l(l+1)}{r^2}\psi(r)=E\psi(r). \quad (239)$$

The corresponding algebraic transformations for Schrödinger equations for both variable energy $E$ and orbital angular momentum $l$ and fixed $E$ and $l$ are obtained naturally as special cases for a definite choice of the function $h(r)$, which is regular except at the point $r=0$.

### 10.1. Darboux transformations

We shall use the technique presented in Ref. 74. We shall seek a solution of Eq. (1) with some initially unknown potential $V(R)$ in terms of the known solutions of Eq. (1) with known potential $V^0(R)$ in the same form:

$$\phi(r,E,\lambda)=y(r)W\{y^0(r),\phi^0(r,E,\lambda)\}, \quad (240)$$

where $W\{y^0(r),\phi^0(r,E,\lambda)\}$ is the Wronskian of the functions $y^0$ and $\phi^0$:

$$W\{y^0(r),\phi^0(r)\}=y^0(r)d\phi^0(r)/dr-dy^0(r)/dr\phi^0(r),$$

as for the ordinary Schrödinger equation (239). However, the functions $y(r)$ and $y^0(r)$ now satisfy Eq. (1) with $V(R)$ and $V^0(R)$, respectively, at some distinguished value $\gamma^2=\gamma'^2$, which may correspond to a bound state. In Eq. (1), $\gamma^2$ is the energy with coefficient $h(r)$ that depends on the coordinate variable. At the same time, the function $h(r)$ must satisfy general requirements imposed on the potential function in scattering theory.[38] Multiplying Eq. (1) for $y^0(r)$ with known potential $V^0(R)$ by $\phi^0(\gamma,r)$, a function for arbitrary $\gamma$, and the equation for $\phi^0(\gamma,r)$ by $y^0(r)$ and subtracting the resulting expressions, we obtain

$$dW(r)/dr=h(r)(\gamma'^2-\gamma^2)y^0(r)\phi^0(\gamma,r). \quad (241)$$

We find the second derivative $d^2\phi(\gamma,r)/dr^2$, using the definition (240) and the relation (241):

$$\frac{d^2\phi(\gamma,r)}{dr^2}=\frac{d^2y(r)}{dr^2}W\{y^0(r),\phi^0(\gamma,r)\}+y(r)\frac{d[h(r)y^0(r),\phi^0(\gamma,r)]}{dr}+2\frac{dy(r)}{dr}h(r)y^0(r)\phi^0(\gamma,r).$$

Taking into account Eq. (1) for $y(r)$ and making the necessary manipulations, we obtain

$$\frac{d^2\phi(\gamma,r)}{dr^2}=[V(r)-\gamma'^2h(r)]y(r)W\{y^0(r),\phi^0(\gamma,r)]+2\frac{d[y(r)y^0(r)]}{dr}h(r)\phi^0(\gamma,r)+\frac{dh(r)}{dr}y(r)y^0\times(r)\phi^0(\gamma,r)+h(r)y(r)W\{y^0(r),\phi^0(\gamma,r)\}.$$

Using the definition (240), we rewrite this relation in the form

$$\left(\frac{d^2}{dr^2}-V(r)+h(r)\gamma^2\right)\phi(\gamma,r)=2h(r)\frac{dy(r)y^0(r)}{dr}\phi^0(\gamma,r)+y(r)y^0(r)\frac{dh(r)}{dr}\phi^0(\gamma,r). \quad (242)$$

It is obvious that the condition of vanishing of the right-hand side of the identity (242),

$$\frac{d}{dr}\ln y(r)y^0(r)=\frac{1}{2}\frac{d}{dr}\ln h(r), \quad (243)$$

leads to the fact that the function $\phi(\gamma,r)$ defined by the relation (240) satisfies Eq. (1). The condition (243) corresponds to

$$y(r)=\frac{1}{\sqrt{h(r)}y^0(r)}. \quad (244)$$

Then, with allowance for the definition (240), the solution of Eq. (1) for arbitrary $\gamma^2$ can be written as follows:

$$\phi(\gamma,r)=\frac{1}{\sqrt{h(r)}y^0(r)}W\{y^0(r),\phi^0(\gamma,r)\}. \quad (245)$$

We now find explicitly an expression for the potential $V(r)$ in terms of the known functions $h(r)$, $y^0(r)$, and $V^0(r)$. We use the relation (244) in Eq. (1) for the function $y(r)$:

$$V(r)=\frac{d^2y(r)/dr^2}{y(r)}+h(r)\gamma'^2=2\left(\frac{dy^0(r)/dr}{y^0(r)}\right)^2-\frac{d^2y^0(r)/dr^2}{y^0(r)}+\frac{dy^0(r)/dr}{y^0(r)}\frac{dh(r)/dr}{h(r)}+h\gamma'^2-\frac{1}{2}\frac{d^2h(r)/dr^2}{h(r)}+\frac{3}{4}\left(\frac{dh(r)/dr}{h(r)}\right)^2.$$

Transforming this expression with allowance for the equations

$$2\left(\frac{dy^0(r)/dr}{y^0(r)}\right)^2-\frac{d^2y^0(r)/dr^2}{y^0(r)}=-2\left(\frac{dy^0(r)/dr}{y^0(r)}\right)'+\frac{d^2y^0(r)/dr^2}{y^0(r)},$$

$$\frac{d^2y^0(r)/dr^2}{y^0(r)}=V^0(r)-h(r)\gamma'^2,$$

we finally obtain

$$V(r)=V^0(r)-2\sqrt{h(r)}\,\frac{d}{dr}\left[\frac{1}{\sqrt{h(r)}}\frac{d}{dr}\ln y^0(r)\right]$$

$$+\sqrt{h(r)}\,\frac{d^2}{dr^2}\frac{1}{\sqrt{h(r)}}. \qquad (246)$$

It is now easy to show how the relations (245) and (246) go over into the corresponding relations[74] for Darboux transformations in the $(\lambda^2,E)$ plane. When the energy $E$ and orbital angular momentum $l$ vary along arbitrary straight lines in the $(\lambda^2,E)$ plane $(\lambda=l+1/2)$, i.e.,

$$aE+bl(l+1)=aE^0+bl^0(l^0+1)=\text{const}, \qquad (247)$$

the radial Schrödinger equation (239) can be written in the form[71]

$$-\left[\frac{d^2}{dr^2}+V(r)+\frac{l^0(l^0+1)}{r^2}-E^0\right]\psi(r,E,l)$$

$$=\gamma^2 h(r)\psi(r,E,l). \qquad (248)$$

Here, $E^0$ and $l^0$ are certain fixed values of the energy and the orbital angular momentum. It is readily verified that then

$$\gamma^2 h(r)=(E-E^0)+\frac{\lambda^{02}-\lambda^2}{r^2}. \qquad (249)$$

Setting

$$h(r)=\frac{a+br^2}{ar^2}, \qquad (250)$$

we immediately obtain from (246) and (245) the analytic relations

$$\phi(r,E,\lambda)=\frac{r}{y^0(r)\sqrt{1+\alpha r^2}}\,W\{y^0(r),\phi^0(r,E,\lambda)\}, \quad (251)$$

$$V(r)=V^0(r)-2\sqrt{\frac{\alpha r^2+1}{r^2}}\,\frac{d}{dr}\left[\sqrt{\frac{r^2}{\alpha r^2+1}}\frac{d}{dr}\ln y^0(r)\right]$$

$$-\frac{3r}{(1+\alpha r^2)^2} \qquad (252)$$

with $\alpha=b/a$. Using (250) and (247) in (249), we can represent $\gamma^2$ either as a function of $E$ or as a function of $\lambda^2$, depending on which of the variables is chosen as independent on the line in the $(\lambda^2,E)$ plane determined by the parameters $a$ and $b$:

$$\gamma^2(E)=(E-E^0)\,\frac{a}{b}, \quad \gamma^2(\lambda)=(\lambda^{02}-\lambda^2). \qquad (253)$$

In the papers of Refs. 69 and 70,

$$h(r)=\frac{a+br^2}{r^2}. \qquad (254)$$

Then, as follows from (249),

$$\gamma^2(E)=(E-E^0)\,\frac{1}{b}, \quad \gamma^2(\lambda)=(\lambda^{02}-\lambda^2)\,\frac{1}{a}.$$

However, the final result does not depend on the choice of $h(r)$ in the form (250), (254) or

$$h(r)=\frac{a+br^2}{br^2},$$

if we require consistency with (240) and (247).

We consider the case of fixed $l$, $\lambda^2=\lambda^{02}$. From the relation (249) we obtain $\gamma^2 h(r)=(E-E^0)$. Comparing this with the first equation of (253), we obtain $h(r)=b/a=\text{const}$. With such $h(r)$, the expressions (246) and (245) for the potential and solutions become

$$V(r)=V^0(r)-2\frac{d^2}{dr^2}\ln y^0(r), \qquad (255)$$

$$\phi(r,E)=\frac{1}{y^0(r)}\,W\{y^0(r),\phi^0(r,E)\} \qquad (256)$$

for $\alpha=\infty$ and the condition $y^0(r)\neq 0$ on the interval $a<r<b$ as obtained from the ordinary Darboux–Crum–Kreĭn transformations. In the case of fixed energy, $E=E^0$, we have $h(r)=1/r^2$. Substituting this $h(r)$ in (246) and (245), we obtain the Darboux transformations

$$V(r)=V^0(r)-\frac{2}{r}\left[\frac{d}{dr}r\frac{d}{dr}\ln y^0(r)\right], \qquad (257)$$

$$\phi(r,\lambda)=\frac{r}{y^0(r)}\,W\{y^0(r),\phi^0(r,\lambda)\}, \quad \lambda\neq\lambda' \qquad (258)$$

for the Schrödinger equation for $E=\text{const}$.

As special cases of a different choice of $h(r)$, we can consider forms of the transformations in the presence of Coulomb forces and a Coulomb coupling constant $C$. If for $l=\text{const}$ we set, for example, $h(r)=(a+br)/r$ under the condition

$$aE+bc=aE^0+bc^0=\text{const},$$

then from the relations (246) and (245) we immediately obtain analytic connecting expressions for the potential and solutions.

Setting $h(r)=r^{-1}$, we obtain the exactly solvable model investigated in Ref. 126 with variable electric charge with fixed angular momentum and energy.

### 10.2. Bargmann transformations

We shall seek solutions of Eq. (1) in the following form, which is more general than (240):

$$\phi(\gamma,r)=\phi^0(\gamma,r)-\sum_{\mu}^{M} y_\mu(r)\,W\{\phi^0(\gamma_\mu,r),\phi^0(\gamma,r)\}, \qquad (259)$$

with functions

$$y_\mu(r)\equiv y(r,\gamma_\mu)=C_\mu\phi(\gamma_\mu,r).$$

We shall find the conditions under which the function $\phi(\gamma,r)$ determined by (259) satisfies Eq. (1). The procedure is analogous to the one proposed in Refs. 73 and 75. We differentiate (259) twice. Taking into account (241), we obtain

$$\frac{d^2\phi(\gamma,r)}{dr^2}=\frac{d^2\phi^0(\gamma,r)}{dr^2}-\sum_{\mu}^{M}\left\{\frac{d^2y_\mu(r)}{dr^2}W\{\phi^0(\gamma_\mu,r),\phi^0(\gamma,r)\}\right.$$

$$+y_\mu(r)\frac{d[h(r)\phi^0(\gamma_\mu,r)\phi^0(\gamma,r)]}{dr}$$

$$\left.+2\frac{dy_\mu(r)}{dr}h(r)\phi^0(\gamma_\mu,r)\phi^0(\gamma,r)\right\}.$$

We transform this expression with allowance for Eq. (1) for $y_\mu(r)$ and the definition (259):

$$\left(\frac{d^2}{dr^2}-V(r)+h(r)\gamma^2\right)\phi(\gamma,r)$$

$$=[-V(r)+V^0(r)]\phi^0(\gamma,r)$$

$$-2\sum_{\mu}^{M}\left\{h(r)\frac{dy_\mu(r)\phi^0(\gamma_\mu,r)}{dr}+y_\mu(r)\phi^0(\gamma_\mu,r)\frac{dh(r)}{dr}\right\}$$

$$\times\phi^0(\gamma,r). \tag{260}$$

The function $\phi(\gamma,r)$ satisfies Eq. (1) if the right-hand side of the last relation vanishes. This condition is equivalent to

$$V(r)=V^0(r)-\sum_{\mu}^{M}\left\{2h(r)\frac{dy_\mu(r)\phi^0(\gamma_\mu,r)}{dr}\right.$$

$$\left.+y_\mu(r)\phi^0(\gamma_\mu,r)\frac{dh(r)}{dr}\right\}. \tag{261}$$

The solution $y_\mu(r)$ with potential (261) can be determined from (259), using the connection $y_\mu(r)=C_\mu\phi(\gamma_\mu,r)$:

$$y_\mu(r)=\sum_{\nu}^{M}C_\nu\phi^0(\gamma_\nu,r)P_{\nu\mu}^{-1}(r), \tag{262}$$

where

$$P_{\mu\nu}(r)=\delta_{\mu\nu}+C_\nu W\{\phi^0(\gamma_\mu,r),\phi^0(\gamma_\nu,r)\}.$$

Substitution of (262) in (261) and (259) enables us to give expressions for the potential and for the solutions corresponding to it in terms of the known function $h(r)$ and known solution $\phi^0(\gamma,r)$:

$$V(r)=V^0(r)-2\sqrt{h(r)}\frac{d}{dr}\left[\frac{1}{\sqrt{h(r)}}\frac{d}{dr}\ln\det P(r)\right], \tag{263}$$

$$\phi(\gamma,r)=\phi^0(\gamma,r)-\sum_{\mu}^{M}\sum_{\nu}^{M}C_\nu\phi^0(\gamma_\nu,r)P_{\mu\nu}^{-1}(r)$$

$$\times W\{\phi^0(\gamma_\mu,r)\phi^0(\gamma,r)\}. \tag{264}$$

It is now easy, using the general expressions (263) and (264) for Eq. (1), to obtain the Bargmann-type transformations

$$V(r)=V^0(r)-2\frac{\sqrt{(1+\alpha r^2)}}{r}\frac{d}{dr}$$

$$\times\left[\frac{r}{\sqrt{(1+\alpha r^2)}}\frac{d}{dr}\ln\det P(r)\right], \tag{265}$$

$$\phi_l(r,E)=\phi_l^0(r,E)-\sum_{\mu}^{M}\sum_{\nu}^{M}C_\nu\phi^0(r,E_\nu,\lambda_\nu)$$

$$\times P_{\nu\mu}^{-1}(r)\frac{W\{\phi^0(r,E_\mu,\lambda_\mu),\phi^0(r,E,\lambda)\}}{E_\mu-E_\nu} \tag{266}$$

for the potentials and solutions (239) and (248) of the Schrödinger equation with variable values of the energy and orbital angular momentum along arbitrary straight lines in the $(\lambda^2,E)$ plane.[73,75] For this, it is sufficient to take $h(r)$ from (250) in (263) and (264). Transformations of Bargmann type were also constructed in Refs. 69 and 70. In the first of these, as in Refs. 73 and 75, algebraic transformations were made, while the second study was based on a general formulation of the inverse problem[71,72] for variable $l$ and $E$. Fixing the angular momentum $l$ or the energy $E$, we can obtain as readily from the relations (263) and (264) the Bargmann expressions for problems with fixed $l$ and $E$, setting $h(r)=b/a$ and $h(r)=1/r^2$, respectively.

The Gel'fand–Levitan or Marchenko approaches with degenerate kernel of the operator of generalized shift $K(r,r')$ can be obtained by a suitable choice of the boundary conditions. In the Gel'fand–Levitan approach developed for regular solutions, the Wronskian in Eqs. (263) and (264) can be expressed as follows:

$$W\{\phi^0(\gamma_\mu,r),\phi^0(\gamma,r)\}$$

$$=(\gamma_\mu^2-\gamma^2)\int_0^r h(r)\phi^0(\gamma_\mu,r')\phi^0(\gamma,r')dr'. \tag{267}$$

In the approach of Marchenko, which uses Jost solutions, the Wronskian is written in the form

$$W\{f^0(\gamma_\mu,r),f^0(\gamma,r)\}$$

$$=(\gamma_\mu^2-\gamma^2)\int_r^\infty h(r)f^0(\gamma_\mu,r')f^0(\gamma,r')dr'. \tag{268}$$

Taking into account the relations (267) or (268) in (264), we obtain

$$\phi(\gamma,r)=\phi^0(\gamma,r)-\sum_{\mu}^{M}\sum_{\nu}^{M}C_\nu\phi^0(\gamma_\nu,r)P_{\nu\mu}^{-1}(r)(\gamma_\mu^2-\gamma^2)$$

$$\times\int_{0(r)}^{r(\infty)}h(r)\phi^0(\gamma_\mu,r')\phi^0(\gamma,r')dr', \tag{269}$$

where we also rewrite $P_{\nu\mu}(r)$ in the integral form

$$P_{\nu\mu}(r)=\delta_{\nu\mu}+C_\mu(\gamma_\mu^2-\gamma^2)$$

$$\times\int_{0(r)}^{r(\infty)}h(r)\phi^0(\gamma_\mu,r')\phi^0(\gamma,r')dr'.$$

It is now obvious that in the relations (259), (262)–(264) we can interpret $\phi$ as all solutions—regular and Jost—of the Sturm–Liouville problem, these in general remaining arbitrary until definite boundary conditions of the problem are chosen.

## 11. CONCLUSIONS

We have formulated the multidimensional inverse scattering problem in the adiabatic representation, and this has necessarily led to the development of two interrelated nonstandard problems: a parametric one, in the fiber for the Hamiltonian of the fast motion, and a multichannel one of gauge type, which describes the slow dynamics of the system. We have proposed a method of constructing a large class of exactly solvable multidimensional models on the basis of the developed technique of Bargmann potentials for the parametric family of inverse problems and for systems of equations with a covariant derivative. The formulation of the parametric inverse problem and the generalization to this case of the technique of Bargmann potentials makes it possible to generate a large class of exactly solvable multidimensional models with specification of a functional dependence on the external coordinate variable of the scattering data.

We have obtained in closed analytic form simple and transparent expressions for the potential matrices of Bargmann type and the solutions corresponding to them, using a factorization with respect to the channel indices of the matrix kernels $Q(X,X')$ and $K(X,X')$ of the equations of the inverse problem, this being in addition to their factorization with respect to the coordinate variable.

Using the expressions given above, we can study some geometrical aspects of scattering theory. For example, specifying the functional dependence of the terms, one can see what is the behavior of the matrix elements of the operator of the induced connection and the scalar potential as the levels approach right up to quasicrossings. In a consistent approach, the parametric dependence of the terms on the slow variables must be determined after solution of the inverse problem for the slow system of equations. In applications, the parametric formulation of the inverse problem in the framework of the adiabatic approach makes it possible to solve effectively the problem of choosing a particular class of spherically asymmetric potentials to be used in calculations of quantum systems.

We have generalized the technique of algebraic transformations for Eq. (1) containing a functional dependence on the coordinate in the term with the energy, in addition to the potential function. In closed form, we have obtained connecting relations for the potentials and the solutions corresponding to them; these generalize the expressions of the Bargmann approach. Special cases of such an approach are transformations with variable and fixed energy, orbital angular momentum, and Coulomb coupling constant.

The relations obtained in the framework of the supersymmetric approach for equations with a gauge potential make it possible to generate a new class of exactly solvable models. Of particular interest is the development of the approach for the construction of models with singular potentials and nontrivial topological phases.

The considered approach opens up new possibilities for the construction of exactly solvable models helpful in the interpretation of the singular behavior of the potential curves; it relates them to the modern geometrical treatment of scattering theory in terms of Hilbert bundles. We should like to point out that in the presence of supersymmetry for systems of gauge equations describing the slow dynamics of quantum-mechanical systems geometric phases arise and topological effects are possible. Nonadiabatic geometric phases, in addition to the Aharanov–Anandan phases, arise from the singularities of the induced connection operator **A** at the points of term crossing.

## APPENDIX

We represent the total wave function in the form

$$|\Psi_j(\mathbf{X},\mathbf{Y})\rangle = \sum_m |\Phi_m(\mathbf{X};\mathbf{Y})\rangle\langle\Phi_m(\mathbf{X};\mathbf{Y})|\Psi_j(\mathbf{X},\mathbf{Y})\rangle$$
$$= \sum_m |\Phi_m(\mathbf{X};\mathbf{Y})\rangle\chi_{mj}(\mathbf{X}) = |\Phi(\mathbf{X};\mathbf{Y})\rangle\chi_j(\mathbf{X}). \quad (270)$$

Note that the total wave function $\Psi$ must be invariant under the unitary transformation

$$\Phi_m(\mathbf{X};\mathbf{Y}) = \sum_n \Phi_n'(\mathbf{X};\mathbf{Y})\,\mathscr{U}_{nm}(\mathbf{X}),$$
$$\chi_m(\mathbf{X}) = \sum_n \mathscr{U}^*_{mn}(\mathbf{X})\chi_n'(\mathbf{X}).$$

We use the decomposition of the identity (completeness)

$$I = \sum_m |\Phi_m\rangle\langle\Phi_m|, \quad (271)$$

and also the orthogonality (integration over the fast variables **Y**)

$$\langle\Phi_m(\mathbf{X};\mathbf{Y})|\Phi_n(\mathbf{X};\mathbf{Y})\rangle = \delta_{mn} \quad (272)$$

of the functions $\Phi$ in the original multidimensional Schrödinger equation

$$H\Psi(\mathbf{X},\mathbf{Y}) = E\Psi(\mathbf{X},\mathbf{Y}), \quad H = -\Delta + V. \quad (273)$$

We make simple transformations, namely, we express the action of the operator $\Delta_{\mathbf{X}}$ on basis functions that depend parametrically on **X**:

$$\Delta_{\mathbf{X}}\Psi(\mathbf{X},\mathbf{Y}) = \nabla\cdot(\nabla\Psi(\mathbf{X},\mathbf{Y})),$$
$$\nabla|\Psi\rangle = (\nabla|\Psi\rangle)\chi + |\Phi\rangle\nabla\chi, \quad (274)$$
$$\nabla(\nabla|\Psi\rangle) = (\nabla^2|\Phi\rangle)\chi + 2\nabla|\Phi\rangle\nabla\chi + |\Phi\rangle\nabla^2\chi.$$

We multiply (273) and (274) from the left by $\langle\Phi_m(\mathbf{X};\mathbf{Y})|$ and integrate over **Y**:

$$\langle\Phi_m|\nabla^2|\Psi\rangle = \langle\Phi_m|(\nabla^2|\Phi_n\rangle)\chi + 2\langle\Phi_m|\nabla|\Phi_n\rangle\nabla\chi$$
$$+ \langle\Phi_m||\Phi_n\rangle\nabla^2\chi, \quad (275)$$

$$\nabla[\langle\Phi_m(\mathbf{X};\mathbf{Y})|\nabla|\Phi_n(\mathbf{X};\mathbf{Y})\rangle]$$
$$= \nabla[\langle\Phi_m(\mathbf{X};\mathbf{Y})]|\nabla|\Phi_n(\mathbf{X};\mathbf{Y})\rangle$$
$$+ \langle\Phi_m(\mathbf{X};\mathbf{Y})|(\nabla^2|\Phi_n(\mathbf{X};\mathbf{Y})\rangle). \quad (276)$$

Using the orthogonality relation (272),

$$\nabla[\langle\Phi_m(\mathbf{X},\mathbf{Y})|\Phi_n(\mathbf{X},\mathbf{Y})\rangle] = \nabla\delta_{mn} = 0,$$

we obtain

$$(\nabla\langle\Phi_m(\mathbf{X,Y})|)|\Phi_n(\mathbf{X,Y})\rangle = -\langle\Phi_m(\mathbf{X,Y})|\nabla|\Phi_n(\mathbf{X,Y})\rangle, \tag{277}$$

i.e., the vector matrices, defined as $i\mathbf{A}_{mn} = -\langle\Phi_m(\mathbf{X,Y})|\nabla|\Phi_n(\mathbf{X,Y})\rangle$, are anti-Hermitian.

Using (277) and (271) in (276), we obtain

$$-i\nabla\mathbf{A}_{nm} = \mathbf{A}_{nm}^2 + \langle\Phi_m(\mathbf{X,Y})|(\nabla^2|\Phi_n(\mathbf{X,Y})\rangle).$$

After this, (275) can be rewritten in terms of the extended derivative in the form

$$-i(\nabla\mathbf{A})\chi - \mathbf{A}^2\chi - 2iA\nabla\chi + \nabla^2\chi = D^2\chi = (\nabla - i\mathbf{A})^2\chi.$$


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